Coalition Formation in the Airport Problem

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1. Motivation

We study the failure of the Pareto axiom and thus the hypothesis that it is the grand coalition that forms in the Airport Problem, when costs are shared according to the Shapley value.

Definition. A finite cooperative game is a pair $G = (N, v)$ in which $N = \{1, 2, ..., n\}$ is the finite set of players and $v : 2^N \to \Re$ is the characteristic function such that $v(\emptyset) = 0$.

In cooperative game theory literature, one of the widely accepted allocation rules is the Shapley value, suggested by Shapley in 1953. This solution concept is axiomatized by four properties efficiency, additivity, symmetry, and dummy player property. Shapley suggests the value $\Phi_i(v)$ to player i in a cooperative game (N, v) as:

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}[v(S \cup \{i\}) - v(S)]$$

This is a weighted average of all possible marginal contributions of player i. What we are interested in is that if in the Airport Problem the Shapley value is the accepted rule of distributing the worth of any coalition among its members, will grand coalition form or not. If not, which coalitions will form?

2. Airport Problem

Suppose n different types of airplanes as players. Since airplanes are different in size and characteristics, the landing bands which they need to land on are different and so, the cost of bands construction would be different. We assume that it costs C_i to airplane type i to build the band alone and if two airplanes have the same cost, we consider them as the same type. For simplicity we assume that there is just one plane of each type. Without loss of generality we can assume $C_1 < C_2 < ... < C_n$. One of the assumptions in the Airport Problem is that if coalition $S = \{(i_1, i_2, ..., i_k)\}, i_j \in \{1, 2, ..., n\}$ such that $i_1 < i_2 < ... < i_k$ forms and builds the band, all players of type $j, j \leq i_k$ can use the band built by this coalition but no player $i > i_k$ can use it. So, if coalition S forms and build the band it can offer the use of the band to all players of type $j \leq i_k$ which do not participate in S and charge all of them the total amount C_m in which $m = \max\{j \mid j < i_k, j \notin S\}$.
The standard cooperative game in this problem is that if $N = \{1, 2, ..., n\}$ for each $S \subseteq N$ the cost for coalition S to build the band is:

$$c(S) = \max\{C_j \mid j \in S\}$$

in which $c(S)$ is the characteristic cost function. Obviously,

$$c(N) = C_n$$

The saving game related to this cost structure is $G = (N, s)$ in which:

$$s(S) = \sum_{j \in S} C_j - c(S)$$

Littlechild and Owen (1973) have shown that the Shapley value assigns the cost y_i to player i by:

$$y_i = y_{i-1} + \frac{C_i - C_{i-1}}{n - i + 1}$$

in which $y_0 = C_0 = 0$. Notice that this allocation rule is under the hypothesis that grand coalition would form.

3. Coalition Formation

The process of forming a coalition is as follows: Any player $i \in N$ can propose a coalition S if $i \in S$. A coalition will form if all its members agree on forming it. Due to the characteristic of Airport Problem, if $n \notin S$, in the case that coalition S forms and build the band, by sure another band would be built for the use of player n. So, if $n \notin S$ and S forms, it would not be the only producer of the band. So, coalition S can not charge other players who do not cooperate with S the amount C_m and the charging amount would diminish because if one of the two producer charges the amount a for an specific type of airplane, the other would suggest $a - \varepsilon$ to the same type as the marginal cost of using the band is zero, so the previous one would suggest $a - 2\varepsilon$ and so on. As a result, in order that coalition S forms and it can charge all other types $j < i_k, j \notin S$, we should have $n \in S$.

4. Model

We claim that in the Airport Problem, if coalitions form freely as in open membership method, the Shapley value allocation rule does not lead to form the grand coalition. To prove this claim, we consider a society in which the Shapley value is accepted as the rule of distributing the worths of coalitions among their members or allocating the cost of building a product among those who use it. We also assume that in this society, it is accepted that after producing the service by a coalition, each member of it would get proportional of the cost that he has paid to produce the service from what the coalition may earn. We show that in this society grand coalition does not form.

Definition. We call a coalition **stable** if the cost of using the service produced by that coalition for all its members is minimized when they join that coalition rather than forming any other coalition.

5. Example

$C_i = i; i \in \{1, 2, ..., n\}$.

We consider coalitions with two members. If player n forms a coalition with player $n - 1$, they can potentially charge other players $n - 2$ units. As we assumed, they will distribute the benefit of $n - 2$ proportional to the amount that they have paid. Due to simplified Shapley value formula player k should pay Φ_k to produce the service as:

$\Phi_n^{\{n,n-1\}} = 1 + \frac{n-1}{2} = \frac{n+1}{2}$

$\Phi_{n-1}^{\{n,n-1\}} = \frac{n-1}{2}$

Net cost of using the service will be:

$NC(n)^{\{n,n-1\}} = \frac{n+1}{2} - \frac{n+1}{2}.\frac{n-2}{n} = \frac{n+1}{n}$

$NC(n-1)^{\{n,n-1\}} = \frac{n-1}{2} - \frac{n-1}{2}.\frac{n-2}{n} = \frac{n-1}{n}$

If player n forms a coalition with player $k, k \in \{1, 2, ..., n - 2\}$ we will have:

$\Phi_n^{\{n,k\}} = n - k/2$

$\Phi_k^{\{n,k\}} = k/2$

So, the net cost of using the service for player n and k would be:

$NC(n)^{\{n,k\}} = (n - \frac{k}{2}).(1 - \frac{n-1}{n}) = (n - \frac{k}{2}).\frac{1}{n}$

$NC(k)^{\{n,k\}} = \frac{k}{2}.(1 - \frac{n-1}{n}) = \frac{k}{2n}$

Player n minimizes his net cost over all players k. So, the minimum net cost which player n should pay would be:

$$(n - \frac{n-2}{2}).\frac{1}{n} = \frac{n+2}{2n}$$

To find the partner who minimizes the net cost of using the service, player n compares $\frac{n+2}{2n}$ and $\frac{n+1}{n}$ and decides to form the coalition with player $n - 2$ as his net cost would be $\frac{n+2}{2n}$ which is absolutely less than $\frac{n+1}{n}$. So, coalition $\{n, n - 2\}$ forms.
One may think of that it may decrease the cost for player n to produce the service alone and then charge others. In this case he should pay the cost of one unit which is more than $\frac{n+2}{2n}$ for all $n > 2$. As n increases player n has more incentive to cooperate with player $n - 2$ as his net cost of using the band decreases more.

6. Theorem

Theorem. In the Airport Problem, if the Shapley value is the rule of allocating the cost of a coalition among its members, the grand coalition would not form. The unique stable two member coalition which forms would be $\{n, n - 2\}$.

References

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