The Dynamics of Closeness and Betweenness[a]

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[a] This poster summarizes Chapter 3 of my thesis. A working paper version can be found in [1].

What would happen if everybody were calculative in building relationships?
We combine two streams of literature by considering beneficial network positions derived in social network analysis (see, e.g., [2]) and game theoretic models of network formation (see, e.g., [3]). Applications are R&D networks, personal (business) contacts, trade among countries, et cetera.

1. Some Notation

Consider a set of *agents* $N = \{1, ..., n\}$ and the set of all undirected *networks* on those agents $G = \{g : g \subseteq g^N\}$, where g^N stands for the complete network. $l_i(g)$ denotes the number of links agent i has in network g. Let $d_{ij}(g)$ be the *distance* (that is the length of a shortest path) between agents i and j. We define the distance of two unconnected agents as $M(> n-1)$.
Closeness measures the access to resources by having short distances. Formally, the closeness of agent i is the reverse average distance (normalized into $[0, 1]$):

$$CLOSE_i(g) = \frac{M}{M-1} - \frac{\sum_{j \in N} d_{ij}(g)}{(M-1)(n-1)}.$$

Betweenness measures intermediation rents by lying on many shortest paths between other agents. Formally,

$$BETW_i(g) = \frac{2}{(n-1)(n-2)} \sum_{j<k(j \neq i, k \neq i)} \frac{\tau^i_{jk}(g)}{\tau_{jk}(g)},$$

where $\tau_{jk}(g)$ is the number of shortest paths between j and k, and $\tau^i_{jk}(g)$ indicates the number of shortest paths between j and k that go through i; the fraction $\frac{\tau^i_{jk}(g)}{\tau_{jk}(g)}$ is replaced by zero, when $\tau_{jk}(g) = 0$.

2. The Centrality Model

Suppose that agents strive for closeness and betweenness, while maintaining links is costly, i.e.

$$u_i(g) = (1-\lambda)CLOSE_i(g) + \lambda BETWEEN_i(g) - cl_i(g).$$

The parameters $(\lambda, c) \in [0, 1] \times \mathbb{R}_+$ stand for the cost of one link and the weight of betweenness versus closeness benefits. Consider a society $(N, G, (u_i)_{i \in N})$ where agents randomly meet to revise their relationships:

1. Start with some network g_0.
2. Pick a pair of agents $\{i, j\}$ at random.
3. Form the link, $g_1 = g_0 \cup ij$, if both improve their utility (at least one strictly). Cut the link, $g_1 = g_0 \setminus ij$, if at least one improves strictly. $g_1 = g_0$, otherwise.
4. Take g_1 and go back to step 1. Stop when no more changes occur.

A necessary condition for a "state of equilibrium" in this society is pairwise **stability**:

(i) $\forall ij \in g, \quad u_i(g) \geq u_i(g \setminus ij)$ and $u_j(g) \geq u_j(g \setminus ij)$

(ii) $\forall ij \notin g, \quad u_i(g \cup ij) > u_i(g) \Rightarrow u_j(g \cup ij) < u_j(g)$.

Fig. 1 illustrates the three methods we use to analyze this model.

1. Equilibrium analysis, i.e. to describe regions of parameters, where certain networks are stable.
2. Enumeration: A computer finds all stable networks along the red lines.
3. Simulation: Let the dynamic process above run for certain settings (red dots).

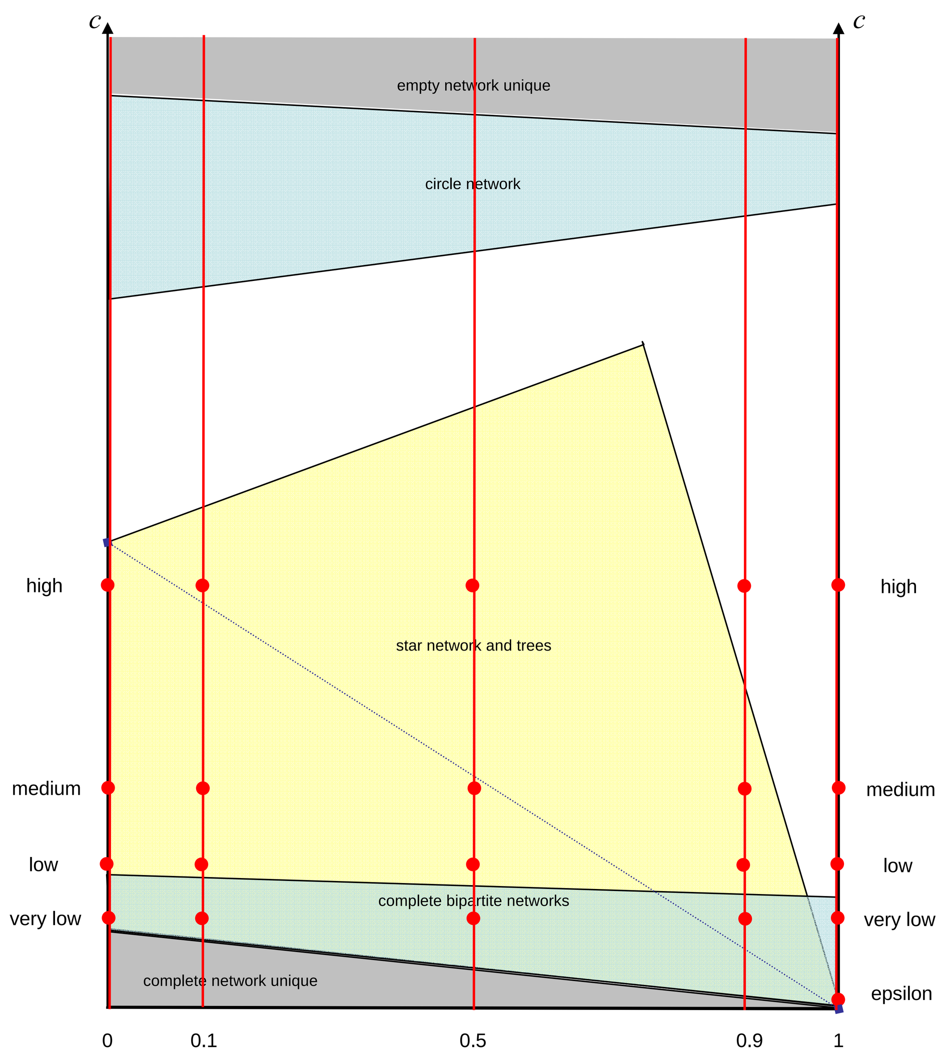

Figure 1: *Parameter "map" (λ horizontally).*

3. Dynamics of Closeness

Consider $\lambda = 0$. Fig. 2 shows the most frequently emerging network (for $c = high$ and $n = M = 8$).

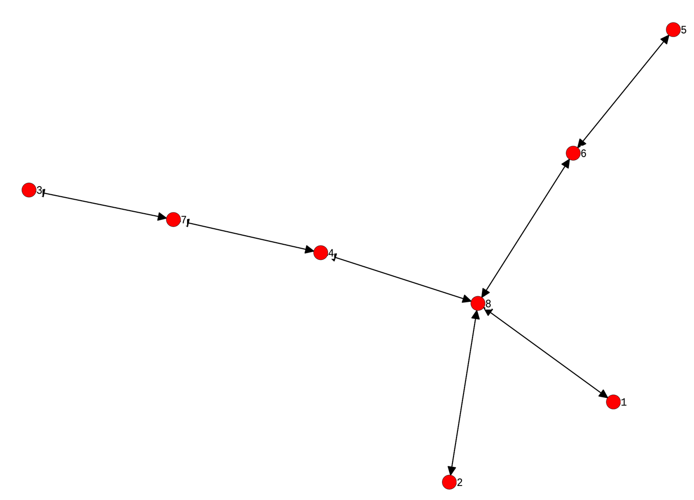

Figure 2: *Typical network for closeness incentives.*

The network exhibits three noteworthy characteristics: It is sparse, connected, and distances are not extremely small. With each method, we were able to show that these characteristics are typical for the dynamics of closeness. E.g. the enumeration shows that 42% of the stable networks are tree graphs, while only 0.2% percent of all possible networks belong to this class.

4. Dynamics of Betweenness

Consider $\lambda = 1$. Fig. 3 shows the most frequently emerging network (for $c = low$ and $n = M = 8$).

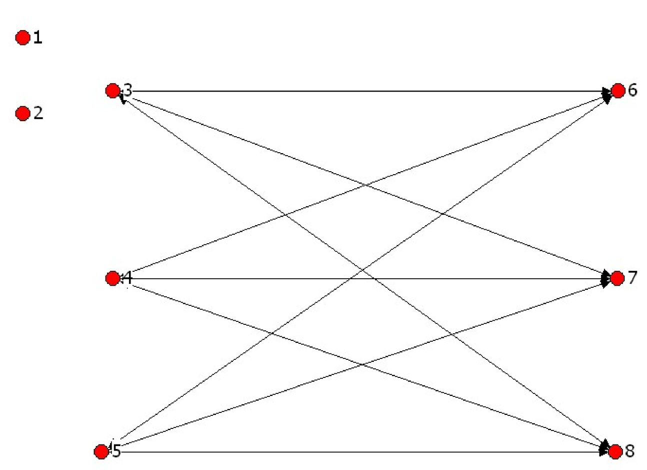

Figure 3: *Typical network for betweenness incentives.*

This network consists of one dense component with small distances. The next table presents how often this type of network (complete bipartite component, with at least two agents in each group, besides only isolates) emerges in the simulation with $n = 8$. Notably for higher costs ($c = med$ and $c = hi$), the empty network emerges in more than 90% of the simulation runs.

	epsilon costs	very low costs	low costs
Freq. of Emergence	40.4%	78.3%	61.1%

5. Interaction

While the results of the last two sections are in line with expectations from literature on similar models, the interaction of multiple incentives, i.e. closeness and betweenness, has not been studied. Fig. 4 depicts the number of stable networks found by enumeration. The networks are organized by the range of weights λ for which they are stable. E.g. for $n = 8$ there are 12,346 non-isomorphic networks; for pure betweenness incentives ($\lambda = 1$) 37 of them can be stable, eight of which can be stable for any λ.

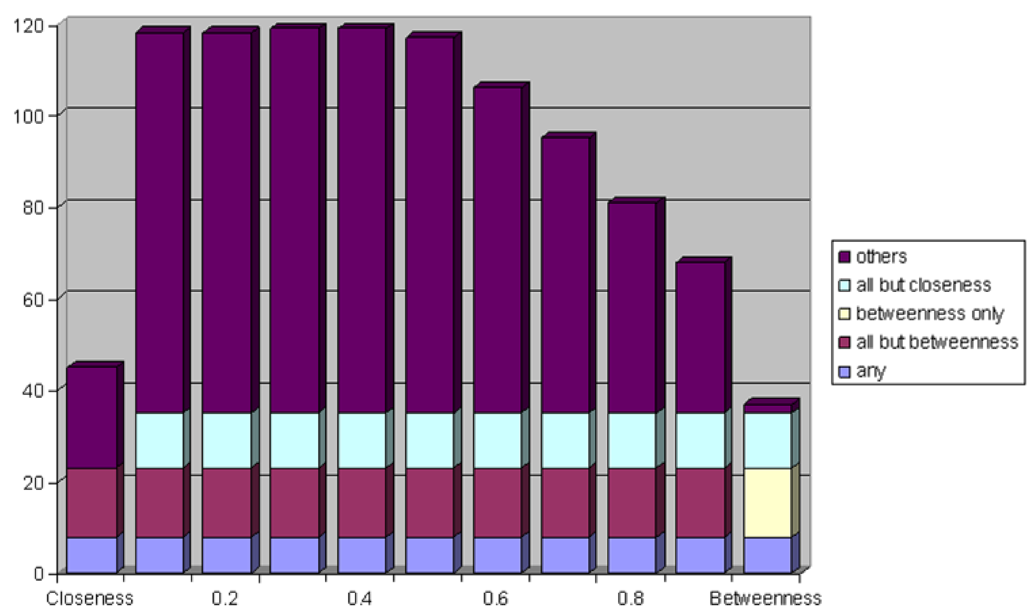

Figure 4: *Number of stable networks.*

Surprisingly, there are many stable networks for mixed incentives ($\lambda = 0.1, ..., 0.9$), that are neither stable for closeness ($\lambda = 0$) nor for betweenness incentives ($\lambda = 1$). This phenomenon results from the interaction between different incentives.

6. Network Closure

The work of [4] has drawn attention to a structural feature of many social networks: Small groups of agents are heavily linked among themselves such that "a friend of a friend is very likely to be my friend".

Proposition 1 *(i) Let a clique of size q be a group of q fully connected agents. For $\lambda = 0$, a network with a clique of size $q \geq 3$ or larger is not stable if $c > \frac{n}{q(M-1)(n-1)}$.*

(ii) For any network g, agent i with $l_i(g) \geq 2$ and costs $c > \frac{1-\lambda}{l_i(g)(M-1)}$, the following holds: if all neighbors of i are directly linked ($CC_i(g) = 1$), then the network is not stable.

This result leaves only a "small corridor in the parameter map" for the stability of networks with high closure. Finally, we assess the closure of the emerging networks with a common transitivity index, namely, the proportion of complete triads among the triads with two or three links. The following table shows the simulation results for $n = M = 14$, while the index of the starting networks is above 50%.

weight c	very low costs	low costs	medium costs
Closeness ($\lambda = 0$)	0.19%	0.19%	0
$\lambda = 0.5$	4.86%	0.02%	0
Betweenness $\lambda = 1$	0.49%	0	0

We learn that linking incentives based on closeness and betweenness interact in non-trivial ways but induce clear patterns in the network structure.

References

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[3] Jackson, M. (2004): A Survey of Models of Network Formation: Stability and Efficiency. In: Demange, G., and Wooders, M. (eds.), Group Formation in Economics; Networks, Clubs and Coalitions. Cambridge University Press.

[4] Watts, D.J., and Strogatz, S.H. (1999): Collective Dynamics of "Small World" Networks. Nature 393, 440-442.