Does it pay to have a balanced government budget?

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Abstract

This paper presents an endogenous growth model with public capital and public debt. The government finances productive and unproductive public spending through income taxation and through public deficits. In addition, the primary surplus to GDP ratio is set such that it is a positive function of the debt ratio which is a necessary condition for the inter-temporal budget constraint of the government to be fulfilled. The paper then studies growth and welfare effects of the model assuming a balanced government budget and compares the outcome to the scenario where public debt grows in the long-run, but at a smaller rate than capital and consumption, and to the scenario where public debt grows at the same rate as capital and consumption. The analysis is undertaken both for the model on the balanced growth path as well as for the model on the transition path.

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1 Introduction

One strand in the endogenous growth literature assumes that the government invests in a productive public capital stock which raises the incentive to invest (see e.g. Futagami et al., 1993). This approach goes back to Arrow and Kurz (1970) who, however, do not analyze models leading to sustained growth endogenously.

Most of the endogenous growth models with productive public spending are characterized by the assumption of a balanced government budget. Exemptions of this are the approaches by Greiner and Semmler (2000) and by Ghosh and Mourmouras (2004). In these contributions it is assumed that the government may finance public expenditures by deficits but the government has to stick to some well-defined budgetary regimes. Greiner and Semmler (2000) study growth effects of fiscal policy and find that more strict regimes generate a higher balanced growth rate because the debt ratio in these regimes is smaller compared to that in less strict budgetary regimes, where the government may run deficits not only to finance public investment. Ghosh and Mourmouras (2004) analyze welfare effects of these regimes and demonstrate that the choice of the budgetary regime does not only affect the long-run growth rate but is also crucial as concerns welfare. An interesting contribution along this line of research is provided by Futagami et al. (2006) who study an endogenous growth model with productive public spending and public debt but assume that government debt must converge to a certain exogenously given debt ratio asymptotically. They demonstrate that there exist two balanced growth paths to which the economy can converge in the long-run, with one being saddle point stable and the other being saddle point stable or asymptotically stable. Further, these authors show that a deficit financed increase in productive public spending raises the low balanced growth rate while it reduces the high balanced growth rate.

While the assumption of budgetary regimes or of a debt ratio to which an economy must converge in the long-run is plausible and can be found in the real world, it may nevertheless be considered as ad hoc. However, this does not hold for the inter-temporal budget constraint of the government. This constraint is in a way a natural constraint any government must obey. A contribution where the inter-temporal budget constraint of the government is taken into account is the paper by Neil and Turnovsky (1999). These authors study the question of when a reduction in the income tax rate, alone or together with a decline of government expenditures, improves the long-run fiscal balance of the government. Another example is the paper by Greiner (2007). In that model it is assumed that the primary surplus of the government is a positive linear function of the debt to gross national income ratio which guarantees that the inter-temporal budget constraint of the government is fulfilled. The paper, then, analyzes the question of how much of the available tax revenue must be used for the debt service so that sustained growth is possible as well as the dynamic effects of a deficit financed increase in public investment. In addition, the paper demonstrates that the reaction of the primary surplus to higher government debt is crucial as concerns the dynamics and that the economy may converge to stable limit cycles and not necessarily to a balanced growth path.

The assumption that the primary surplus positively depends on public debt can also be observed in the real world. Bohn (1998), for example, finds evidence for the USA that the debt ratio positively affects the primary surplus to GDP ratio. Greiner et al. (2007) study countries of the EURO area and also find empirical evidence for a positive dependence of the primary surplus to GDP ratio on the public debt to GDP ratio. Therefore, integrating this assumption in a theoretical model seems to be justified.

It should also be pointed out that, given a fixed tax rate and fixed unproductive public spending, a rise in the primary surplus ratio, as a result of a higher debt ratio, can lead to a decline in productive public spending.[1] But there is empirical evidence that public investment is indeed reduced as the debt service rises, instead of other unproductive public spending. Examples of such studies are the ones by Oxley and Martin (1991), Gong et al. (2001) or Heinemann (2002). The fact that an increase in the primary surplus, as

[1]Taking the tax smoothing rule seriously, taxes should indeed be constant over time (cf. Barro, 1979).

debt grows, guarantees sustainability of public debt, is more general than the Ricardo equivalence theorem and contains the latter as a special case. This holds because there are three ways for the government to raise the primary surplus: First, it can raise taxes, second it can reduce spending and, third, the surplus can increase due to a higher GDP leading to more tax revenues.

In this paper we take up the approach by Greiner (2007) and try to answer the question of whether a balanced budget yields a better outcome in terms of growth and welfare compared to a scenario where public debt grows in the long-run, but at a smaller rate than the balanced growth rate, and compared to a scenario where public debt grows at the same rate as all other variables. This question is of high relevance for policy makers but has not been dealt with rigourously in the economics literature, as far as we know.

The rest of the paper is organized as follows. In the next section, we present the structure of our model. Section 3 analyzes our model where we first study the asymptotic behaviour and then analyze growth and welfare effects of different debt scenarios and section 4, finally, concludes.

2 The structure of the growth model

Our economy consists of three sectors: A household sector which receives labour income and income from its saving, a productive sector and the government. First, we describe the household sector.

2.1 The household

The household sector is represented by one household which maximizes the discounted stream of utility arising from per-capita consumption, $C(t)$, over an infinite time horizon subject to its budget constraint, taking factor prices as given. The utility function is assumed to be logarithmic, $U(C) = \ln C$, and the household has one unit of labour, L, which it supplies inelastically.[2] The maximization problem can be written as

$$\max_{C} \int_{0}^{\infty} e^{-\rho t} \, \ln C \, dt, \tag{1}$$

subject to

$$(1-\tau)\,(w + rW + \pi) = \dot{W} + C. \tag{2}$$

ρ is the subjective discount rate, w is the wage rate and r is the interest rate. $W \equiv B+K$ gives wealth which is equal to public debt, B, and private capital, K, and π gives possible profits of the productive sector, the households takes as given in solving its optimization problem. Finally, $\tau \in (0,1)$ is the constant income tax rate. The dot gives the derivative with respect to time and we neglect depreciation of private capital.

To solve this problem we formulate the current-value Hamiltonian which is written as

$$\mathcal{H} = \ln C + \gamma((1-\tau)\,(w + rW + \pi) - C) \tag{3}$$

Necessary optimality conditions are given by

$$C^{-1} = \gamma \tag{4}$$

$$\dot{\gamma} = \rho\gamma - \gamma(1-\tau)\,r \tag{5}$$

If the transversality condition $\lim_{t\to\infty} e^{-\rho t} W/C = 0$ holds, which is fulfilled for a time path on which assets grow at the same rate as consumption, the necessary conditions are also sufficient.

2.2 The productive sector

The productive sector is represented by one firm which behaves competitively and which maximizes static profits. The production function of the firm is given

$$Y = K^{1-\alpha} G^{\alpha} L^{\epsilon}, \tag{6}$$

[2] From now on we omit the time argument t if no ambiguity arises.

with $\alpha + \epsilon \leq 1$. $(1-\alpha)$ is the private capital share and ϵ gives the labour share. G denotes public capital and α gives the elasticity of output with respect to public capital. Using that labour is set to one, $L = 1$, profit maximization gives

$$w = \epsilon K^{1-\alpha} G^{\alpha} \tag{7}$$

$$r = (1-\alpha) K^{-\alpha} G^{\alpha} \tag{8}$$

2.3 The government

The government in our economy receives tax revenues from income taxation and has revenues from issuing government bonds it then uses for public investment, I_p, and for public consumption, C_p. As concerns public consumption we assume that this type of spending does neither yield utility nor raise productivity but is only a waste of resources. Further, the government sets the primary surplus such that it is a positive linear function of public debt which guarantees that public debt is sustainable. In order to see this, we note that the accounting identity describing the accumulation of public debt in continuous time is given by:

$$\dot{B} = rB(1-\tau) - S, \tag{9}$$

where S is government surplus exclusive of net interest payments.

The inter-temporal budget constraint of the government is fulfilled if

$$B(0) = \int_0^{\infty} e^{-\int_0^{\mu}(1-\tau)r(\nu)d\nu} S(\mu) d\mu \leftrightarrow \lim_{t\to\infty} e^{-\int_0^{t}(1-\tau)r(\mu)d\mu} B(t) = 0 \tag{10}$$

holds. Equation (10) is the present-value borrowing constraint which states that public debt at time zero must equal the future present-value surpluses.

Now, assume that the ratio of the primary surplus to GDP ratio is a positive linear function of the debt to GDP ratio and of a constant. The primary surplus ratio, then, can be written as

$$\frac{S}{Y} = \phi + \beta \, \frac{B}{Y} = \frac{\tau Y - I_p - C_p}{Y}\,, \tag{11}$$

where $\phi \in \mathbb{R}$, $\beta \in \mathbb{R}_{++}$ are constants. The parameter β determines how strong the primary surplus reacts to changes in public debt and ϕ determines whether the level of the primary surplus rises or falls with an increase in GDP.

Using (11) the differential equation describing the evolution of public debt can be written as

$$\dot{B} = (r(1-\tau) - \beta)\, B - \phi\, Y. \tag{12}$$

Solving this differential equation and multiplying both sides by $e^{-\int_0^t (1-\tau) r(\mu) d\mu}$ to get the present-value of public debt leads to

$$e^{-\int_0^t (1-\tau) r(\mu) d\mu} B(t) = e^{-\beta t}\, B(0) - \phi\, Y(0)\, \frac{\int_0^t e^{\beta\mu}\, e^{-\int_0^\mu ((1-\tau) r(\nu) - g_Y(\nu)) d\nu} d\mu}{e^{\beta\, t}}\,. \tag{13}$$

with $B(0) > 0$ public debt at time $t = 0$ and g_Y the growth rate of GDP.

Equation (13) shows that $\beta > 0$ is necessary for $\lim_{t\to\infty} e^{-\int_0^t (1-\tau) r(\mu) d\mu} B(t) = 0$. Further, if the numerator in the second expression in (13) remains finite the second term converges to zero. If the numerator in the second expression in (13) becomes infinite, l'Hôpital gives the limit as $e^{-\int_0^t ((1-\tau) r(\nu) - g_Y(\nu)) d\nu} / \beta$. This shows that $\beta > 0$ and $\lim_{t\to\infty} \int_0^t ((1-\tau) r(\nu) - g_Y(\nu)) d\nu = \infty$ are sufficient for sustainability of public debt. It should be noted that for t sufficiently large, $(1-\tau) r - g_Y > 0$ always holds in our model because we assume a logarithmic utility function and because the growth rate of GDP converges to the balanced growth rate.

These considerations demonstrate that a positive linear dependence of the primary surplus to GDP ratio, i.e. $\beta > 0$, is a necessary condition for the inter-temporal budget constraint of the government to be met. Therefore, we posit that the government sets the primary surplus according to (11) so that the evolution of public debt is given by (12).

Defining $C_p/I_p = \kappa$ as public consumption relative to public investment and using that the evolution of public debt is given by $\dot{B} = rB(1-\tau) + I_p(1+\kappa) - \tau Y = rB(1-\tau) - \beta B - \phi Y$ public investment can be written as

$$I_p = \omega(\tau - \phi) Y - \omega\beta B\,, \tag{14}$$

where $\omega = 1/(1+\kappa)$. Neglecting depreciation, the differential equation describing the evolution of public capital, then, is written as

$$\dot{G} = I_p = \omega(\tau - \phi)Y - \omega\beta B\,. \tag{15}$$

2.4 Equilibrium conditions and the balanced growth path

Before we analyze our model we give the definition of an equilibrium and of a balanced growth path. An equilibrium allocation for our economy is defined as follows.

Definition 1 *An equilibrium is a sequence of variables* $\{C(t), K(t), G(t), B(t)\}_{t=0}^{\infty}$ *and a sequence of prices* $\{w(t), r(t)\}_{t=0}^{\infty}$ *such that, given prices, the firm maximizes profits, the household solves (1) subject to (2) and the budget constraint of the government (9) is fulfilled with the primary surplus set according to* (11)*.*

Resorting to (4), (5) and (8), the growth rate of consumption is derived as

$$\frac{\dot{C}}{C} = -\rho + (1-\tau)(1-\alpha)K^{-\alpha}G^{\alpha}. \tag{16}$$

The economy-wide resource constraint is obtained by combining (12) and (2) as

$$\frac{\dot{K}}{K} = -\frac{C}{K} + \frac{K^{1-\alpha}G^{\alpha}}{K} + \beta\,\frac{B}{K} + (\phi - \tau)\,\frac{K^{1-\alpha}G^{\alpha}}{K}\,. \tag{17}$$

Thus, in equilibrium the economy is completely described by equations (16), (17), (12) and (15) plus the limiting transversality condition of the household.

In definition 2 we define a balanced growth path.

Definition 2 *A balanced growth path (BGP) is a path such that the economy is in equilibrium and such that consumption, private capital and public capital grow at the same strictly positive constant growth rate, i.e.* $\dot{C}/C = \dot{K}/K = \dot{G}/G = g$, $g > 0$, $g = constant$, *and either*

(i) $\dot{B} = 0$ *or*

(ii) $\dot{B}/B = g_B$, *with* $0 < g_B < g$, $g_B = constant$, *or*

(iii) $\dot{B}/B = \dot{C}/C = \dot{K}/K = \dot{G}/G = g$.

Definition 2 shows that we consider three different scenarios. Scenario (i) is the balanced budget scenario where the government has at each point in time a balanced budget. But this does not necessarily imply that public debt equals zero. If the level of initial debt is positive, the debt to capital ratio and also the debt to GDP ratio are positive but decline over time and converge to zero in the long-run. Scenario (ii) describes a situation where the government always runs a deficit so that the growth rate of public debt is positive in the long-run. But public debt grows at a smaller rate than capital, consumption and output.[3] This implies that the debt ratio also converges to zero in the long-run since public debt grows at a smaller rate than capital and output. The last scenario, scenario (iii) finally, describes the case which is characterized by public deficits where government debt grows at the same rate as all other endogenous variables in the long-run.

To analyze our economy around a BGP we define the new variables $x \equiv G/K$, $b \equiv B/K$ and $c \equiv C/K$. Differentiating these variables with respect to time leads to a three dimensional system of differential equations given by

$$\dot{x} = x\left((\tau-\phi)x^{\alpha-1}\omega - \omega\beta b/x + c - x^{\alpha} - \beta b + (\tau-\phi)x^{\alpha}\right), \tag{18}$$

$$\dot{b} = b\left((1-\alpha)x^{\alpha}(1-\tau) - \beta - \phi x^{\alpha}/b + c - x^{\alpha} - \beta b + (\tau-\phi)x^{\alpha}\right), \tag{19}$$

$$\dot{c} = c\left((1-\alpha)x^{\alpha}(1-\tau) - \rho + c - x^{\alpha} - \beta b + (\tau-\phi)x^{\alpha}\right). \tag{20}$$

A solution of $\dot{x} = \dot{b} = \dot{c} = 0$ with respect to x, b, c gives a BGP for our model and the corresponding ratios $x^{\star}, b^{\star}, c^{\star}$ on the BGP.[4] In the next section we analyze growth and welfare effects of our scenarios as given in definition 2.

3 Analysis of the model

In this section we study the structure of our model as well as growth and welfare effects of the different scenarios. Further, we analyze how a transition from scenario (iii), where

[3]Of course, GDP grows at the same rate as capital and consumption on a BGP.

[4]The $\star$ denotes BGP values and we exclude the economically meaningless BGP $x^{\star} = c^{\star} = 0$.

the debt ratio is strictly positive in the long-run, to scenario (i) and to scenario (ii), and vice versa, affects growth and welfare.

3.1 The asymptotic behaviour of the model

First, we analyze scenario (i) and scenario (ii). Scenario (i) is obtained by setting the reaction coefficient β equal to the net return on capital, $(1-\tau)r$, making β an endogenous variable. Further, ϕ is set equal zero for all times, i.e. $\phi = 0$, for $t \in [0, \infty)$. Scenario (ii) is obtained by setting $\phi = 0$ and by letting β be an exogenous parameter which can take arbitrary but strictly positive values. Proposition 1 gives results as concerns existence, uniqueness and stability of a balanced growth path for these two scenarios.

Proposition 1 *There exists a unique saddle point stable balanced growth path for scenario (i). For $\rho < \beta < r(1 - \tau)$, scenario (ii) is also characterized by a unique saddle point stable balanced growth path.*

Proof: See appendix.

This proposition demonstrates that both the balanced budget scenario and the scenario with public deficits but an asymptotically zero debt ratio are characterized by unique BGPs which are saddle point stable, where a certain parameter restriction must be fulfilled for scenario (ii). The restriction $\rho < \beta < r(1-\tau)$ states that, on the one hand, β must not be too small, $\beta > \rho$, so that sustained growth is possible. This holds because otherwise public debt would become too large requiring too many resources for the debt service so that ongoing growth would not be possible. The positive effect of β on the growth rate can be seen from equation (17). On the other hand, β must not be too large, $\beta < r(1-\tau)$, because otherwise the government would not invest enough in public capital so that sustained growth would not be possible either, which can be seen from (15).

Saddle point stability means that there exists a unique value $c(0)$ such that the economy converges to the balanced growth path in the long-run. If one takes both $x(0)$ and $b(0)$ as given, since both x and b are state variables, this implies that the economy is determinate. However, from an economic point of view, it seems plausible to make a difference between capital stocks and public debt. This holds because capital stocks need a longer time period to be built up whereas public debt can be changed faster since it is a financial variable. Therefore, from an economic point of view the assumption that $b(0)$ can be controlled could also be justified.

As concerns scenario (iii), where public debt grows at the same rate as consumption and capital in the long-run, the analytical model turns out to be quite complicated and no unambiguous results can be derived.[5] But it is possible to derive a result as concerns the public debt to private capital ratio for the analytical model. This is the contents of proposition 2.

Proposition 2 *Assume that there exists a balanced growth path in scenario (iii). Then, the ratio of public debt to private capital is given by*

$$b^\star = \frac{\omega\,(\tau-\phi)(x^\star)^\alpha - g\,x^\star}{\beta\,\omega}\,.$$

$\phi < \tau$ *is necessary for* $b^\star$ *to be positive and* $\phi \geq \tau$ *is sufficient for* $b^\star$ *to be negative.*

Proof: See appendix.

Proposition 2 shows that the reaction of the primary surplus to variations of GDP is crucial as concerns the question of whether sustained growth is feasible in the long-run together with a positive value of public debt. It should be recalled that the parameter ϕ determines whether, and if so how strong, the level of the primary surplus rises as GDP increases. Proposition 2 states that for relatively large values of ϕ, i.e. for $\phi \geq \tau$, sustained growth is only feasible if public debt is negative, that is if the government is a creditor. At first sight, this result may seem counter intuitive. However, if the government puts too high a weight on controlling public debt, by setting ϕ to a high value, it spends too little for public investment so that in this case sustained growth is only feasible if the government has built up a stock of wealth out of which it finances productive public

[5] In Greiner (2007) it is shown that the dynamics crucially depends on the values of β and ϕ.

spending. From a technical point of view, this is seen from the differential equation for $\dot{G}$, equation (15), which shows that public investment would be negative for $\phi \geq \tau$, unless B was negative, too.

In the next subsection we analyze growth effects of the different scenarios.

3.2 Growth effects of fiscal policy

Our first concern is to answer the question of which scenario brings about a higher growth rate of consumption and of capital in the long-run. Proposition 3 gives the answer to this question.

Proposition 3 *Assume that the government does not dispose of a stock of wealth and that there exists a balanced growth path with a strictly positive public debt ratio in scenario (iii). Then, the balanced growth growth rate in scenario (iii) is lower than the balanced growth rate in scenario (i). Further, the balanced growth rate in scenario (i) is equal to the balanced growth rate in scenario (ii).*

Proof: See appendix.

The outcome that the balanced growth rate in scenario (i), the balanced budget scenario, is equal to that obtained in scenario (ii), where public debt debt grows less than capital and output in the long-run, is not too surprising. This holds because asymptotically the debt ratio equals zero in both scenarios, so that both scenarios are described by the same equations.

A more interesting result is the outcome that a balanced budget always leads to a higher growth rate in the long-run compared to a scenario where public debt grows at the same rate as consumption and capital. This is indeed a strong result because it states that, starting from a balanced budget, deficit financed public investment can never raise the long-run growth rate if it leads to a positive debt ratio in the long-run. The economic intuition behind this result is that a positive debt ratio in the long-run requires resources for the debt service which cannot be used for productive public spending, leading to a lower balanced growth rate. Hence, starting from a balanced government budget, a deficit financed increase in public investment raises the transitional growth rates of private and public capital but brings about a lower growth rate in the long-run, unless the government balances its budget again or lets public debt grow at a smaller rate than GDP in the long-run, so that the debt to GDP ratio converges to zero.

The only possibility to achieve a balanced growth rate exceeding the one of the balanced budget scenario is given if the government has built up a stock of wealth it uses to finance its expenditures and to borrow to the household sector. In this case, the government is a creditor implying that b is negative.[6] In a corollary to proposition 3 we treat this case.

Corollary 1 *Assume that the government has built up a stock of wealth. Then, the balanced growth rate in scenario (iii) exceeds the balanced growth rate of scenario (i).*

Proof: See appendix.

It must be pointed out that the result in proposition 3 only states that a balanced government budget, or a scenario where public debt grows at a smaller rate than capital and output in the long-run, gives a higher balanced growth path compared to a scenario where public debt grows at the same rate as capital and output. It does not say anything about long-run growth effects of deficit financed public investment given the scenario where public debt grows at the balanced growth rate in the long-run. Thus, a deficit financed increase in public debt may yield a higher balanced growth rate in the latter scenario.

In order to see this we resort to a numerical example. As to the parameter values we set the elasticity of output with respect to public capital to 20 percent, i.e. $\alpha = 0.2$. The income tax rate is set to 10 percent, $\tau = 0.1$, and the rate of time preference is 15 percent,

[6] If b is negative, $|b| < 1$ must hold because otherwise total wealth of the household would be negative which would not make sense.

$\rho = 0.15$. Assuming that one time period comprises several years such a high rate can be justified. The reaction coefficient β is set to $\beta = 0.05$ and $\omega = 0.1$.

Table 1 gives the balanced growth rate, g, and the debt to private capital ratio on the BGP, $b^\star$, for different values of ϕ.

	$\phi = 0.005$	$\phi = 0.0025$	$\phi = -0.0025$	$\phi = -0.005$
g	0.189	0.191	0.196	0.198
$b^\star$	0.024	0.012	-0.012	-0.024

Table 1: Balanced growth rate and the debt to private capital ratio for different ϕ.

To interpret the outcome shown in table 1, it should be noted that a deficit financed increase in public investment is modelled by a decline in ϕ which can be seen from (14). Thus, table 1 demonstrates that a deficit financed increase in public investment raises the balanced growth rate. One can also realize that for negative values of ϕ, implying that the primary surplus declines as GDP rises, sustained growth is only feasible with a negative public debt with this parameter constellation. In this case, the government must have built up a stock of wealth out of which it finances productive public spending.

It should also be mentioned that the Jacobian matrix of (18)-(20), with the parameter values underlying table 1, is characterized by one negative eigenvalue and two positive eigenvalues. Thus, the government must be able to control initial public debt, for example by levying a lump-sum tax at $t = 0$, and set $b(0)$ such that the economy starts on the one-dimensional stable manifold leading the economy to the BGP in the long-run.

Before we study welfare effects, we illustrate the effect resulting from a transition from scenario (iii), where public debt grows at the balanced growth rate in the long-run, to the balanced budget scenario, scenario (i). To do so, we assume that the economy is originally on the BGP when the government decides to balance its budget from $t = 0$ onwards. To do so, we choose the parameter values underlying table 1 with $\phi = -0.005$ but set β to a higher value, $\beta = 0.25$, so that the $b^\star$ is positive and the Jacobian matrix of (18)-(20) has two negative eigenvalues.

To analyze the effects of a change from scenario (iii) to scenario (i) we study the solution of the linearized system of (18)-(20) which is given by

$$x(t) = x^\star + C_1\, v_{11}\, e^{\lambda_1 t} + C_2\, v_{21}\, e^{\lambda_2 t}, \tag{21}$$

$$b(t) = b^\star + C_1\, v_{12}\, e^{\lambda_1 t} + C_2\, v_{22}\, e^{\lambda_2 t}, \tag{22}$$

$$c(t) = c^\star + C_1\, v_{13}\, e^{\lambda_1 t} + C_2\, v_{23}\, e^{\lambda_2 t}, \tag{23}$$

with v_{jl} the $l-$th element of the eigenvector belonging to the negative real eigenvalue λ_j, $j = 1, 2$. C_j, $j = 1, 2$, are constants determined by the initial conditions $x(0)$ and $b(0)$. Setting $t = 0$ gives C_j, $j = 1, 2$, as a function of $x(0)$ and $b(0)$. Inserting these C_j, $j = 1, 2$, in (23) gives the unique $c(0)$ on the stable manifold leading to the BGP in the long-run. Given $x(t)$, $b(t)$ and $c(t)$ from (21)-(23) one can compute the growth rates of C, B, G and K according to (16), (12), (15) and (17).

Figure 1 shows the transitional growth rates C, G and K after the government balances its budget at time $t = 0$.

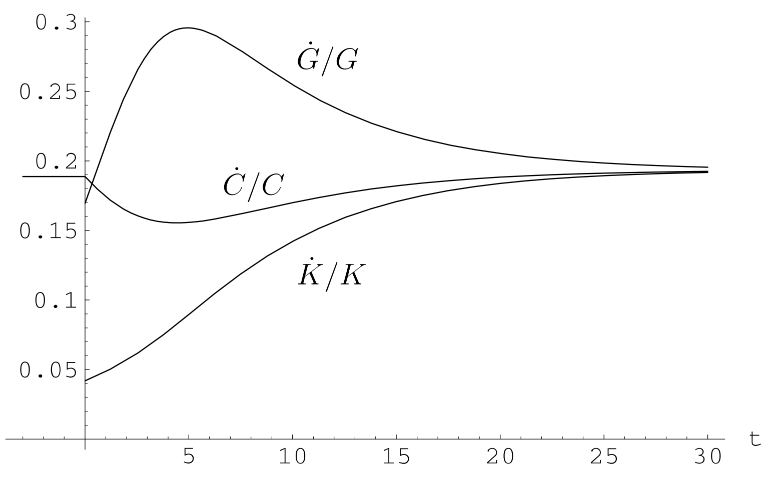

Figure 1: Transitional growth rates of consumption, private capital and public capital after a transition from scenario (iii) to scenario (i) at $t = 0$.

The constant line left of the $t = 0$ axis gives the balanced growth rate of the economy in scenario (iii) which is $g = 0.189$. Switching to a balanced government budget at $t = 0$, by setting $\phi = 0$ and $\beta = (1 - \tau)r$, leads to a downward jump of the growth rate of public capital at $t = 0$ because ϕ is increased from $\phi = -0.005$ to $\phi = 0$ and β also rises at $t = 0$ which has a negative effect on public investment, which can be seen from (15). Hence, balancing the government budget at $t = 0$ brings about an immediate reduction in public investment, which was to be expected. Since x is fixed at $t = 0$ the growth rate of consumption does not react at $t = 0$. The growth rate of private capital jumps downward at $t = 0$ because the ratio of consumption to private capital rises and compensates the increase in ϕ and β. The growth rate of public debt, of course, equals zero from $t = 0$ onward. Over time consumption, private capital and public capital converge to the balanced growth rate of scenario (i) given by $g = 0.193$.

For sake of completeness we want to mention that a change from scenario (iii) to scenario (ii), where public debt grows at a smaller rate than capital in the long-run, gives the same picture as shown in figure 1 from a qualitative point of view. That is, the growth rate of public capital first jumps down, then rises and overshoots its long-run value before it converges to the balanced growth rate. The growth rate of private consumption declines and then rises again. The private capital stock first jumps down and then rises again and converges to the BGP. But there is a difference in the adjustment path of public debt. Since ϕ rises from $\phi = -0.005$ to $\phi = 0$ the growth rate of public debt jumps down at $t = 0$, but since β remains unchanged it is not immediately equal to zero as it is the case when the government switches to the balanced budget scenario. The growth rate of public debt continues to decline for a certain time period and, then, rises again and converges to its new long-run value which is smaller than the growth rates of consumption and capital.

We also want to state that a change from the balanced budget scenario, scenario (i), to scenario (iii), where public debt grows at the same rate as all other variables in the long-run, is just reverse to figure 1. Thus, both private and public investment jump upwards at $t = 0$ and then converge to the BGP, where the growth rate of public capital overshoots, or better undershoots, its long-run value. The growth rate of consumption rises and then declines again and converges to the balanced growth rate. The growth rate of public debt, which equals zero for $t < 0$, jumps upward at $t = 0$ and then declines again and also approaches the balanced growth rate.

Finally, it should be pointed out that the change from one scenario to another scenario is only possible if the BGP values in the old scenario, which are the initial conditions for the new scenario, do not differ too much from the BGP values of the new scenario. For example, setting $\phi = -0.015$ would not allow to switch from scenario (iii) to scenario (i) or to scenario (ii). In this case, the government would have to gradually adjust its spending policy by reducing ϕ step by step before it can switch to the balanced budget scenario for example.

In the next subsection we study welfare effects of fiscal policy for our model.

3.3 Welfare analysis

It is well known that growth and welfare maximization are different goals in the endogenous growth model with a productive public capital stock (cf. Futagami et al., 1993). Therefore, we study welfare effects of fiscal policy in this subsection.

In particular, we are interested in three policy experiments. First, we study the question of whether switching from scenario (iii), where the debt ratio is positive in the long-run, to scenario (i) and to scenario (ii) raises welfare. Second, we analyze welfare effects of switching from a balanced budget scenario, scenario (i), to scenario (iii), where the government either builds up a stock of wealth or runs into debt. Third, we calculate welfare for the three scenarios for given initial conditions with respect to the capital stocks and with respect to public debt.

To compute welfare effects we numerically calculate the expression

$$F = \arg\max_{C(t)} \int_0^{t_f} e^{-\rho t} \ln C(t) dt \,, \tag{24}$$

where t_f denotes the final period and where we set $K(0) = 1$. The value for consumption is obtained by numerically solving equation (16), with $x(t)$ again given by (21)-(23).

In table 2 we report the outcome of our first policy experiment. As to the parameter values we use those of the last section with $\beta = 0.25$, in scenario (ii) and (iii), and $\phi = -0.005$ in scenario (iii).

	$t_f = 1$	$t_f = 5$	$t_f = \infty$
scenario (iii)	-1.247	-3.592	-1.174
from scenario (iii) to (ii)	-0.767	-1.880	1.350
from scenario (iii) to (i)	-0.793	-2.001	1.209

Table 2: Welfare in scenario (iii) and welfare resulting from a transition to scenario (ii) and to scenario (i), respectively.

The first column in table 2 gives welfare F computed according to (24) for scenario (iii) on the BGP for $[0, t_f]$. Column 2 and column 3 give welfare for $[0, t_f]$ when the government switches from scenario (iii) to scenario (ii) and (i), respectively, at $t = 0$.

It can be realized that scenario (iii), where debt grows at the balanced growth rate, leads to smaller welfare than a transition from scenario (iii) to scenario (ii), where debt grows in the long-run but at a lower rate than capital and output, and to smaller welfare than a transition to scenario (i), the balanced budget scenario. Comparing scenarios (ii) and (i), one can realize that scenario (i), the balanced budget scenario, yields lower welfare than scenario (ii), where public debt grows in the long-run but less than output.

The reason for this outcome is that consumption at $t = 0$ in scenario (i) rises less than in scenario (ii). On the other hand, the growth rate of consumption in scenario (ii) temporarily declines and the decline is stronger than that in scenario (i). But this transitionally higher growth rate of consumption in scenario (i), compared to scenario (ii), is not sufficient to compensate for the stronger increase of consumption at $t = 0$ in scenario (ii). Therefore, scenario (ii) yields higher welfare than scenario (i) as reported in the table above.

In table 3 we present the outcome of our second policy experiment where the government switches from a balanced budget scenario, scenario (i), to scenario (iii) with a negative government debt and with a positive government debt, respectively. Technically, this is achieved by setting $\phi = 0.005$ giving a debt ratio on the BGP of $b^\star = -0.0241$, in the first case, and by setting $\phi = -0.005$ giving $b^\star = 0.0235$ in the second case.

	$t_f = 1$	$t_f = 5$	$t_f = \infty$
scenario (i)	-1.254	-3.592	-1.037
from scenario (i) to (iii) with $b^\star < 0$	-0.767	-1.845	1.600
from scenario (i) to (iii) with $b^\star > 0$	-2.361	-7.702	-8.293

Table 3: Welfare in scenario (i) and welfare resulting from a transition to scenario (iii) with $b^\star < 0$ and $b^\star > 0$, respectively.

Analogously to table 2, the first column in table 3 gives welfare F computed according to (24) for scenario (i) on the BGP for $[0, t_f]$. Column 2 and column 3 give welfare for $[0, t_f]$ when the government switches from scenario (i) to scenario (iii) at $t = 0$ with a negative and positive government debt on the BGP, respectively.

Table 3 shows that switching from a balanced budget scenario to a scenario where the government builds up a stock of wealth raises welfare. The reason for this outcome is that the government surplus at $t = 0$ leads to downward jump of public and private investment[7] and to a rise in the level of consumption at $t = 0$. The growth rate of consumption declines temporarily but the decline is compensated by the increase in consumption at $t = 0$ and by the higher balanced growth rate so that welfare rises for all $t \geq 0$. If the government

[7]The growth rates on the transition path are qualitatively the same as those shown in figure 1.

switches to scenario (iii) with a positive debt in the long-run, the effects are just reverse. Now, the deficit financed public investment brings about an increase in both public and private investment leading to an upward jump of these variables, while consumption is reduced at $t = 0$. Although the growth rate of consumption rises temporarily, welfare declines for all t because of the initial decrease in consumption and because of the smaller balanced growth rate.

In the last experiment, finally, we set the initial conditions with respect to x and b to arbitrary values and, then, compute welfare for the three scenarios. Table 4 gives the result of this exercise with $x(0) = 0.03$ and $b(0) = 0.02$.

	$t_f = 1$	$t_f = 5$	$t_f = \infty$
scenario (i)	-0.879	-2.203	1.202
scenario (ii)	-0.858	-2.109	1.317
scenario (iii)	-1.420	-4.150	-1.921

Table 4: Welfare in scenario (i), in scenario (ii) and in scenario (iii) for given initial conditions $x(0) = 0.03$ and $b(0) = 0.02$.

Table 4 partly confirms the outcome of table 2. It demonstrates that scenario (iii), where public debt grows at the balanced growth rate in the long-run, performs worse than scenario (i) and worse than scenario (ii), independent of the time horizon. Comparing the balanced budget scenario, scenario (i), with scenario (ii), where public debt grows less than capital and consumption in the long-run, shows that for this example scenario (ii) always leads to higher welfare than scenario (i), in contrast to table 2, where the balanced budget scenario performed better for a sufficiently large time horizon. The different outcome is due to the different initial conditions with respect to x and b.

Hence, the main conclusion we can draw from this subsection is that scenario (iii), where public debt grows at the balanced growth rate in the long-run, performs worse than scenario (i) and worse than scenario (ii) as concerns welfare. Comparing scenario (i) with scenario (ii) shows that scenario (ii) seems to perform better. The reason for this result is that initial private consumption in the balanced budget scenario, scenario (i), is smaller than in scenario (ii), where debt grows but less than output. However, since the difference is only small care must be taken in generalizing this result. This holds because it cannot be excluded that the initial conditions of the capital stocks and of government debt may be decisive as to whether scenario (i) or scenario (ii) performs better.

4 Conclusion

In this paper we have presented an endogenous growth model with productive public spending and a public debt where the government can run deficits in order to finance public investment. In addition, we have posited that the primary surplus to GDP ratio is a positive function of the debt ratio because this guarantees that public debt remains sustainable. The main results of analyzing our model can be summarized as follows.

1. It turned out that a balanced budget scenario brings about a higher long-run growth rate than a scenario where public debt grows at the balanced growth rate, i.e. at the same rate as all other variables. Further, a scenario where public debt grows in the long-run, but at a lower rate than the balanced growth rate, yields the same long-run growth rate as the balanced budget scenario. A balanced growth rate exceeding the one of the balanced budget scenario can only be obtained when the government is a creditor. This means that the government must have built up a stock of wealth it uses to finance its expenditures and to lend to the private sector.

2. Starting from a balanced budget, a deficit financed public investment raises transitional growth rates but leads to a smaller long-run growth rate if this fiscal policy leads to a positive debt ratio in the long-run. Only if the government switches back to the balanced budget scenario or to the scenario where public debt grows slower than capital and output, a temporarily deficit financed public investment raises transitional growth

without leading to smaller growth in the long-run.

3. The fact that a balanced budget gives the highest possible growth rate in the long-run, unless the government is a creditor, does not imply that a deficit financed increase in public investment always reduces long-run growth. Thus, if the economy is on a balanced growth path where public debt grows at the balanced growth rate, a deficit financed increase in public investment may lead to a higher balanced growth path. However, in this case the model is unstable implying that the government must impose an additional lump-sum tax to control public debt.

4. As concerns welfare, numerical examples have shown that the scenario where public debt grows at the balanced growth rate yields smaller welfare than the balanced budget scenario and smaller welfare than the scenario where public debt grows at a smaller rate than capital and output in the long-run. Further, evidence was found that the latter scenario leads to higher welfare than the balanced budget scenario. However, it cannot be excluded that this result depends on the initial conditions with respect to the capital stocks and with respect to public debt. Here, additional research is necessary in order to find how robust this outcome is.

Overall, it can be stated that a scenario where public debt grows at the same rate as capital and output yields smaller growth and welfare in the long-run, compared to the balanced budget scenario and compared to the scenario where debt grows, but slower than capital and output. Comparing the latter two scenarios, a balanced budget scenario may perform worse so that the scenario where debt grows, but less than output and capital, makes the economy better off compared to the balanced budget scenario. But even that scenario would require in part drastic changes for countries of the EURO area where many economies have difficulties in sticking to the 60% debt criterion. In any case, changing policies such that debt ratios decline over time instead of remaining constant would benefit economies.

Appendix

Proof of proposition 1

To prove this proposition with scenario (i), we set $\phi = 0$, $\beta = (1-\tau)(1-\alpha)x^{\alpha}$ and $b = 0$. Setting $\dot{x} = 0$ and solving this equation with respect to c gives c as a function of x and of parameters. Substituting this function for c in $\dot{c}$ gives $q(x,\cdot) = (1-\alpha)x^{\alpha}(1-\tau) - \rho - \omega\tau x^{\alpha-1}$. It is easily seen that $\lim_{x\to 0} q(x,\cdot) = -\infty$, $\lim_{x\to\infty} q(x,\cdot) = +\infty$ and $\partial q(\cdot)/\partial x > 0$. Thus, existence of a unique BGP is shown.

To show saddle point stability, we compute the Jacobian matrix evaluated at the rest point of (18)-(20). The Jacobian is given by

$$\mathrm{J} = \begin{bmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial b & \partial\dot{x}/\partial c \\ 0 & \partial\dot{b}/\partial b & 0 \\ \partial\dot{c}/\partial x & \partial\dot{c}/\partial b & \partial\dot{c}/\partial c \end{bmatrix}.$$

One eigenvalue of this matrix is given by $\lambda_1 = \partial\dot{b}/\partial b = -\dot{K}/K = -g$. Thus, we know that one eigenvalue, λ_1, is negative. Further, it is easily shown that $(\partial\dot{x}/\partial x)(\partial\dot{c}/\partial c) - (\partial\dot{x}/\partial c)(\partial\dot{c}/\partial x) < 0$ holds, so that complex conjugate eigenvalues are excluded. The determinant of J is given by $\det \mathrm{J} = \partial\dot{b}/\partial b(\tau(\alpha-1)x^{\alpha-2}\omega - (1-\tau)(1-\alpha)\alpha x^{\alpha-1})c^{\star}x^{\star} > 0$. Since the product of the eigenvalues equals the determinant, $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det \mathrm{J} > 0$, and because of $\lambda_1 < 0$, we know that two eigenvalues are negative and one is positive.

For scenario (ii) we set $\phi = 0$ and $b = 0$. Then, we proceed analogously so that existence and uniqueness is readily shown. For $\dot{B}/B < \dot{C}/C$ to hold we must have $\rho < \beta$ and $\beta < (1-\tau)r$ must hold for $\dot{B}/B > 0$. Because of $b^{\star} = 0$ the Jacobian matrix is the same as for scenario (i) except for $\partial\dot{b}/\partial b$. $\partial\dot{b}/\partial b$ now is given by $\partial\dot{b}/\partial b = \lambda_1 = \dot{B}/B - \dot{K}/K < 0$, because of $\dot{B}/B < \dot{K}/K$ at the BGP. In particular, the determinant is again positive implying that two eigenvalues are negative and one is positive. □

Proof of proposition 2

To prove proposition 2, $\dot{c} = 0$ is solved with respect to c giving $c = c(x, b, \cdot)$. Inserting $c = c(x, b, \cdot)$ in $\dot{x}$ and solving $\dot{x} = 0$ with respect to b gives $b^\star$ as shown in proposition 2. It is immediately seen that $\phi < \tau$ is a necessary condition for $b^\star$ to be positive while $\phi \geq \tau$ is a sufficient condition for $b^\star$ to be negative. □

Proof of proposition 3

To prove this proposition we note that we set $\phi = 0$, $\beta = (1-\tau)(1-\alpha)x^\alpha$ and $b = 0$ to get scenario (i). Further, the balanced growth rate is given by $\dot{C}/C = -\rho + (1-\tau)(1-\alpha)x^\alpha$. Along a BGP we have $\dot{C}/C = \dot{G}/G$ which implies

$$-\rho + (1-\tau)(1-\alpha)x^\alpha = \tau\,\omega\,x^{\alpha-1} \qquad (A.1)$$

The left hand side (l.h.s.) in $(A.1)$ is monotonically increasing in x and the right hand side (r.h.s.) is monotonically declining in x. A value $x_i^\star$ such that the l.h.s. in $(A.1)$ equals the r.h.s. gives a BGP for scenario (i).

For scenario (ii), $\dot{C}/C = \dot{G}/G$ implies

$$-\rho + (1-\tau)(1-\alpha)x^\alpha = \tau\,\omega\,x^{\alpha-1} - \phi\,\omega\,x^{\alpha-1} - \omega\,\beta\,b/x \qquad (A.2)$$

Again, a value $x_{ii}^\star$ such that the l.h.s. in $(A.2)$ equals the r.h.s. gives a BGP for scenario (ii).

The function on the l.h.s. of equation $(A.1)$ and of equation $(A.2)$ are identical. The graph of the function on the r.h.s. of $(A.1)$, however, is above the graph of the function on the r.h.s. of $(A.2)$ for all $x \in [0, \infty)$ for $b \geq 0$ and for $\phi > 0$. Therefore, the l.h.s. and the r.h.s. in $(A.1)$ intersect at a larger value of x than the l.h.s. and the r.h.s. in $(A.2)$, giving a higher balanced growth rate for scenario (i). To show this for $\phi < 0$, we note that on the BGP $b^\star$ is given by $b^\star = \phi(x^\star)^{\alpha-1}/(\rho - \beta)$ which follows from $\dot{C}/C = \dot{B}/B$. Inserting this in the r.h.s. of $(A.2)$ and deleting the * gives

$$-\rho + (1-\tau)(1-\alpha)x^\alpha = \tau\,\omega\,x^{\alpha-1} - \phi\,\omega\,x^{\alpha-1}\rho/(\rho - \beta) \qquad (A.3)$$

If $-\phi\,\omega\,x^{\alpha-1}\rho/(\rho-\beta) < 0$, the point of intersection of the l.h.s. and the r.h.s. in (A.1) occurs at a larger value of x than in (A.3). For $\beta > \rho$ it is immediately seen that $-\phi\,\omega\,x^{\alpha-1}\rho/(\rho-\beta) < 0$ holds (recall that $\phi < 0$). For $\beta < \rho$ the reverse holds, but public debt becomes negative because of $b^\star/x^\star = \phi(x^\star)^{\alpha-1}/(\rho-\beta)$. In this case, $x^\star$ in (A.3) is larger than $x^\star$ in (A.1) yielding a higher growth rate for scenario (iii) but this occurs only if public debt is negative.

In scenario (ii) the asymptotic public debt ratio equals zero such that (A.1) holds for both scenario (i) and for scenario (ii) implying that the two scenarios yield the same balanced growth rate. □

Proof of corollary 1

To prove this corollary we have to show that $\psi > 0$ implies $b^\star < 0$, with ψ given by $\psi = -\phi\,\omega\,x^{\alpha-1}\rho/(\rho-\beta)$ from the r.h.s. in (A.3) from the proof of proposition 3. This holds because $\psi > 0$ implies that x which solves (A.1) is smaller than that x which solves (A.3), so that the balanced growth rate of scenario (i) is smaller than the balanced growth rate of scenario (iii). From the proof of proposition 3 we know that $b^\star$ on the BGP is given by $b^\star = \phi(x^\star)^\alpha/(\rho-\beta)$. It is immediately seen that $b^\star$ and ψ have the opposite sign since ω and x are positive. □

References

Arrow, K.J. and M. Kurz (1970) *Public Investment, the Rate of Return, and Optimal Fiscal Policy,* The John Hopkins Press, Baltimore.

Barro, R.J. (1979) On the determination of public debt, *Journal of Political Economy*, **87,** 940-71.

Bohn, H. (1998) The behaviour of U.S. public debt and deficits, *Quarterly Journal of Economics*, **113,** 949-63.

Bruce, N. and S.J. Turnovsky (1999) Budget Balance, welfare, and the growth rate: "Dynamic scoring" of the long-run government budget, *Journal of Money, Credit and Banking,* **31,** 162-86.

Futagami, K., Y. Morita and A. Shibata (1993) Dynamic analysis of an endogenous growth model with public capital, *Scandinavian Journal of Economics,* **95,** 607-25.

Futagami, K., T. Iwaisako and R. Ohdoi (2006) Debt policy rule, productive government spending, and multiple growth paths, (mimeo, Ritsumeikan University).

Greiner, A. and W. Semmler (2000) Endogenous growth, government debt and budgetary regimes, *Journal of Macroeconomics,* **22,** 363-84.

Gong, G., A. Greiner and W. Semmler (2001) Growth effects of fiscal policy and debt austainability in the EU, *Empirica*, **28,** 3-19.

Ghosh, S. and I. Mourmouras (2004) Endogenous growth, welfare and budgetary regimes, *Journal of Macroeconomics*, **26,** 363-84.

Greiner, A., Köller U. and Semmler W. (2007) Debt sustainability in the European Monetary Union: Theory and empirical evidence for selected countries, *Oxford Economic Papers,* **59,** 194-218.

Greiner, A. (2007) An endogenous growth model with public capital and sustainable government debt, forthcoming in: *Japanese Economic Review.*

Heinemann, F. (2002) Factor mobility, government debt and the decline in public investment, *ZEW Discussion Paper*, No. 02-19 (http://zew.de/).

Oxley, H. and J.P. Martin (1991) Controlling government spending and deficits; Trends in the 1980s and prosepcts for the 1990s. *OECD Economic Studies,* **17,** 145-89.