# Modelling Equilibrium Play as Governed by Analogy and Limited Foresight*

Philipp C. Wichardt†

*Department of Economics, University of Bonn*

This Version: November 15, 2007

**Abstract** This paper proposes a bounded rationality approach to model equilibrium play in games. It is based on the observation that decision makers often do not seem to fully distinguish between different but seemingly similar decisions. To capture this, for each player a similarity grouping of decisions is defined based on equality of available actions and analogy of locally foreseen subtrees. The considered equilibrium concept is a (trembling-hand) perfect Nash-equilibrium (Selten, 1975) in which players are required to choose the same behaviour for similar decisions. Based on the approach, it is shown how the Chain Store Paradox (Selten, 1978) can be resolved, and how mixed equilibria in the Centipede Game (Rosenthal, 1981) can be rationalised.

Key words: Bounded Rationality, Chain Store Paradox, Imperfect Recall, Limited Foresight, Reasoning by Analogy

JEL code: C72, D83

*Acknowledgements: I am indebted to Philippe Jehiel, who kindly agreed to supervise a great deal of this research, as well as to Avner Shaked, who later helped me with many challenging discussions, for their advice and support (and their patience). Moreover, I am grateful to Anke Gerber, Georg Nöldeke, Clemens Puppe and Reinhard Selten for helpful comments and suggestions. Financial support of the European Community, a Marie Curie scholarship at the University College London, and the German Research Foundation (DFG), through GRK 629 and SFB/TR 15 at the University of Bonn, is gratefully acknowledged. The usual disclaimer applies.

†Postal address for correspondence: Economic Theory 3, University of Bonn, Adenauerallee 24-26, D-53113 Bonn, Germany; e-mail: philipp.wichardt@uni-bonn.de.

# 1 Introduction

Rationality of agents probably is one of the most contentious assumptions made in non-cooperative game theory and perhaps even in economics as a whole. Assuming it, we are often able to analyse decision problems in a tractable way and to obtain clear-cut results. Yet, these theoretical outcomes seldom seem to match the observations commonly made, for example, in lab experiments (e.g. Camerer, 2003). Dropping the assumption of rational agents, however, it is not clear how to fill the emerging gap. How much rationality is appropriate, under which circumstances will it obtain, and how can it be modelled? Answers to these questions so far have mostly proved elusive and the development of satisfactory models of boundedly rational behaviour remains a great challenge (Rubinstein, 1998).

This paper is a behaviouristic attempt to contribute to the research facing this intriguing challenge to model bounded rationality. Inspired by earlier approaches (e.g. Jehiel, 2004; Jehiel and Samet, 2005; Rubinstein, 1998; Selten, 1978), it proposes a formal approach to model equilibrium play of boundedly rational agents in multistage games with almost perfect information and perfect recall. The considered large class of games is interesting as a lot of evidence for boundedly rational behaviour derives from games that fall into this category (see, e.g., Camerer, 2003). Within the model, agents are boundedly rational in that they are assumed to be restricted in their ability (or willingness) to distinguish between a priori different decision nodes, and to lump some of these decision nodes together. Eventually, players then are required to choose the same behaviour whenever to decisions are "similar."

Similarity thereby is meant to reflect general analogy considerations about the respective decision at hand (separating Bach-vs-Stravinsky from Beer-vs-Quiche) as well as analogy considerations about the local strategic environment as far as foreseen by the player (further separating, e.g., repetitions of Bach-vs-Stravinsky form alternations between Bach-vs-Stravinsky with Beer-vs-Quiche). The equilibrium concept eventually employed is that of a (trembling-hand) perfect Nash-equilibrium (cf. Selten, 1975) in which the emerging similarity partition of decisions, referred to as the player's decision partition, replaces the standard information partition. The equilibrium is required to be perfect in order to impose some rationality constraints on decisions off the equilibrium path which are not yet determined via similarity by on-path behaviour. This is desirable as we think of equilibrium as being the result of some unmodelled learning process.

The current approach is closely related to earlier work by Jehiel which is concerned with both the modelling of analogy considerations in the players' decision making and the players' limited ability to foresee the evolution of the game (cf. Jehiel, 1995, 1998, 2001, 2004). Yet, the present discussion differs from this literature in various aspects. In particular, the existing models focus on the consequences of the agents' simplified perception of the other players' strategies. Accordingly, foresight in these models relates to behaviour, i.e. agents make forecasts about the future course of the game. Actions at each node, then, have to be chosen optimally given the expected behaviour of opponents (which in equilibrium is required to be consistent with the actual one). By contrast, the focus of the current approach lies on the modelling of the agent's simplified perception of the decision problem itself. Hence, foresight only relates to the type of decision ahead, but not to behaviour. Moreover, the information about foreseen decisions is only used to differentiate between decisions which a priori had been considered analogous.[1] It is not used to establish any local beliefs about the expected behaviour of the opponents. In equilibrium, players are "simply" required to optimally choose one action for all nodes considered to be similar - irrespective of any immediate consequences. And optimality here is thought of as resulting from some unspecified (long run) learning process rather than the players' current deliberation.

In fact, given that strategies can be viewed as an *automatised* response to similar (local) strategic environments, the intuition behind the current

[1] In a finitely repeated game with all decisions considered analogous, for example, terminal decisions may still be distinguished from earlier ones according to whether termination is foreseen or not.

approach is somewhat reminiscent of finite automata (cf. Rubinstein, 1986, 1998). However, automata respond to observed past behaviour, i.e. their focus is backward looking and on behaviour. By contrast, the main focus of the current approach is forward looking and on the strategic structure of the game not on behaviour; a history dependent analogy grouping of decisions will be considered among the extensions, though. Moreover, different from automata, the current approach is not restricted to repeated games.

Another way to view the current approach is in terms of imperfect recall (Kuhn, 1953). Roughly speaking, imperfect recall refers to a situation where a player at a later stage of a sequential game is unable to recall information that had been accessible to him before, e.g. moves he has made. In the present framework, such situations (technically) arise almost naturally from the considered similarity grouping of decisions. Thus, in contrast to earlier discussions (e.g. Piccione and Rubinstein, 1997; Rubinstein, 1998), imperfect recall in the current study is not part of the set-up but arises as part of the analysis. Moreover, our goal is not to study how behaviour can be optimised given imperfect recall but rather to discuss how situations of imperfect recall may arise (through analogy grouping and the limited foresight) and what the consequences may be in terms of boundedly rational equilibrium behaviour.

Finally, it is interesting to note that the current approach to some extent captures two aspects of the three-level decision theory sketched by Selten (1978) in conjunction with the famous Chain Store Paradox. As Selten argues, each decision can be made at a level of routine - based on standardised decisions, a level of imagination - based on the players' attempt to visualise how the game may evolve, or a level of (rational) reasoning, with higher levels necessitating the help of lower ones.[2] Obviously, the analogy grouping of similar (own) decision nodes can be seen as capturing the routine aspect. Foresight, in turn, allows players to refine the analogy grouping according to their local perception of the game tree and to behave fully rational towards the end of the game. Hence, it can be interpreted as a link between routine, imagination, and rationality. Yet, in particular the intuition of imagination

[2] For a more detailed discussion, see Selten, 1978, pp. 147-152.

can never be fully grasped within our model. To be different from routine, imagination should be flexible and allow for some adjustment or learning. The static nature of the present framework, however, does not allow us to capture such dynamic aspects of individual decision making. Nonetheless, we hope that the reader will be convinced by the subsequent discussion that this partial approach already constitutes an interesting contribution to the modelling of bounded rationality.

The rest of the paper is structured as follows. In Section 2, the main ingredients of the solution concept are illustrated by means of a simple example. A formal definition of the basic concept, then, is provided in Section 3, and is followed by some further illustrating examples in Section 4. In Section 5, two extensions of the basic framework are discussed, namely additional past dependence of the similarity grouping (5.1), and endogenous but costly foresight (5.2). Moreover, it is shown how these extensions allow us to rationalise less paradox equilibria in the Chain Store Game (Selten, 1978), and mixed equilibria in the Centipede Game (Rosenthal, 1981). Section 6 concludes.

# 2 An Introductory Example

Before entering into any technicalities, the simple example presented in this section shall illustrate all relevant building blocks of the approach.

Consider the 2-player 3-stage game $G_1$ with initial move by Nature depicted in Figure 1. In the first stage, Nature chooses with equal probability one of two possible subtrees. In stage two, player 1 has to decide between left (L) and right (R). Player 2, then, is called upon to move in stage 3 and either has to decide between left (l) and right (r) in all cases, or has to decide between up (u) and down (d) at $h_6$ and between l and r everywhere else. Payoffs are as specified in the figure.

The starting point of the present approach is a primary analogy grouping. For this, players group together all decisions with the same available actions; i.e. player 1 groups together $h_1$ and $h_2$ and player 2 groups together $h_3$, $h_4$,

$G_1$:

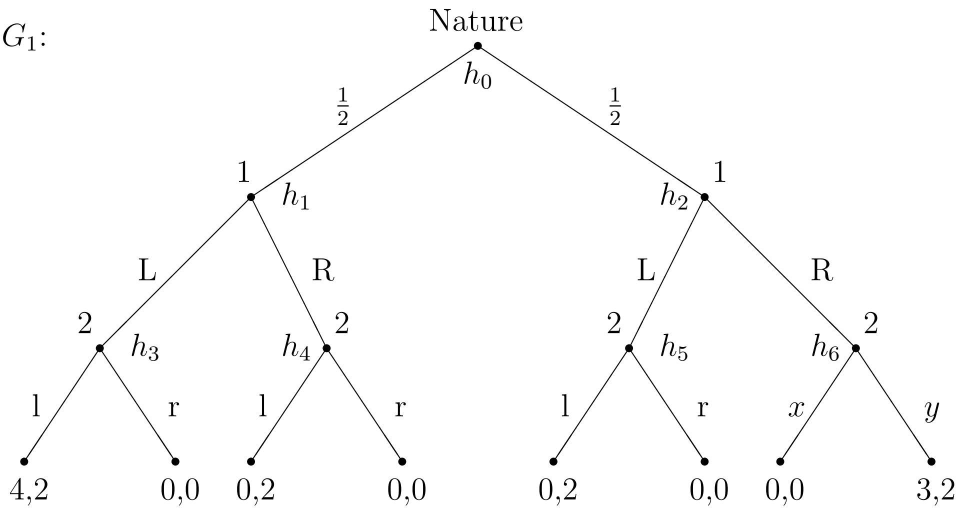


Figure 1: A 3-stage game, where either $(x, y)$=(l,r), or $(x, y)$=(u,d).

and $h_5$ and further adds $h_6$ if and only if $(x, y)$=(l,r).[3] In a second step, this primary analogy grouping is further differentiated if, for some decisions in the same analogy class, equal actions lead to different foreseen types of decisions. To illustrate this point, consider player 1 and assume that he can look exactly one stage ahead. In this case, he will be able to further separate $h_1$ and $h_2$ if and only if $(x, y)$=(u,d). The point to note is that, although L in either case is foreseen to be followed by an l/r-decision of player 2, the foreseen consequences of R are different: if $(x, y)$=(u,d), R at $h_1$ is followed by an l/r-decision of player 2, but R at $h_2$ is followed by a u/d-decision of player 2. Thus, player 1 is able to separate $h_1$ and $h_2$. However, if $(x, y)$=(l,r) no further differentiation is possible, irrespective of the ability to foresee one stage ahead.[4] Eventually, any two nodes that are grouped together after foresight has been added are called *similar*.

In equilibrium, players have to choose the same actions for similar decisions and strategy choices have to be robust to small trembles in the other players' behaviour. For the above example with $(x, y)$=(u,d) and no foresight

[3] If $(x, y)$=(u,d), available actions at $h_6$ are different from the available actions at $h_3$-$h_5$.

[4] This distinction, of course, presumes that labels are not completely arbitrary but are chosen so as to convey differences and similarities between the actual decisions to be made.

this means that (L,ld) is the unique equilibrium as player 1 is constrained to choose the same action at $h_1$ and $h_2$. However, if player 1 can foresee one stage ahead, he is able to further differentiate between $h_1$ and $h_2$ and, hence, (LR,ld) becomes the unique equilibrium of $G_1$. In both cases, all equilibria with player 2 choosing anything but l at $h_3$-$h_5$ and d at $h_6$ are ruled out by requiring equilibria to be perfect; i.e., as usual, imposing perfectness ensures some form of sequential rationality. Moreover, having foresight 1 increases player 1's expected payoff by 1.5 units (from 2 to 3.5). Thus, we would expect player 1 to be willing to invest up to 1.5 payoff-units into having foresight 1, if foresight was a costly matter of choice (cf. Section 5.2).

# 3 The Formal Concept

In the sequel, we formally present the different building blocks of our approach. Defining first the primary analogy grouping of decisions in a game tree, we next move on to the definition of foresight and analogy of foreseen subtrees. Based on these notions, a similarity partition of each player's decisions is introduced which accounts for both the primary analogy grouping (separating decisions with different available actions) and the analogy of foreseen subtrees (further separating decisions with analogous available actions if the foreseen subtrees are not analogous). Players then have to choose the same behaviour whenever two decisions are similar. Finally, equilibrium is defined by requiring strategies to constitute a perfect Nash equilibrium of the resulting game, thereby imposing also some restrictions on off-equilibrium path decisions. We start with some technical preliminaries and notation.

**Preliminaries and General Notation**

The general class of games considered are multistage games with almost perfect information and perfect recall (see Fudenberg and Tirole, 1991, subsection 3.3.2); i.e. simultaneous moves and moves by nature are possible but, at any stage, all preceding moves are assumed to be known to every player. The set of players is denoted by $N = \{1, ..., n\}$. The game tree is denoted by $\Upsilon$. At any stage $k$, the history of $k$ is denoted by $h^k$, with the convention that $h^0 = \emptyset$ for the first stage ($k = 0$). Moreover, the set of player $i$'s available actions at stage $k$, which in general may depend on the history $h^k$, is denoted by $A_i(h^k)$, with $A_i(h^k) = \emptyset$ if player $i$ does not actually face any choice given history $h^k$. The set of pure action profiles $a^k$ available at $h^k$ is denoted by $A(h^k)$, i.e. $A(h^k) = \times_i A_i(h^k)$. The set of all $k$-stage histories, i.e. $k$-tuples of the form $h^k = (a^0, ..., a^{k-1})$, is denoted by $H^k$, the set of all possible histories by $H$, i.e. $H = \bigcup_k H^k$. Moreover, it is convenient to define $H^{k+l}(h^k)$, $l > 0$, to be the set of all histories of stage $k+l$ for which $h^k$ is a proper sub-history, i.e. for each $h \in H^{k+l}(h^k)$ action profiles in the first $k$ stages are determined by $h^k$. Finally, the set of all terminal histories $z$ is denoted by $Z$ and players' preferences over lotteries over outcomes are modelled by a utility function $u_i$. For the sake of argument, we restrict attention to finite games.

**Primary Analogy Grouping**

With the above notation, the players' primary analogy grouping of decisions now can be moddelled as an idiosyncratic grouping of non-terminal histories $h \in H \setminus Z$ into subsets $\alpha_i \subset H \setminus Z$, where $\alpha_i(h)$ denotes the analogy class history $h$ is assigned to by player $i$. The primary grouping takes place on the basis of equality of available actions, i.e. $\alpha_i(h) = \alpha_i(\tilde{h})$ if and only if $A_i(h) = A_i(\tilde{h})$. It shall reflect the fact that, at the very least, players are able to distinguish decisions according to the actions available (which, of course, presumes that the labelling of actions is not completely arbitrary but connected to the type of decision being made). The collection of all player $i$'s analogy classes is denoted by $\mathcal{AP}_i$. Moreover, we define: $\mathcal{AP} := \times_{i \in N} \mathcal{AP}_i$, with elements $\alpha \in \mathcal{AP}$. Slightly abusing notation, we write $A_i(\alpha_i)$ to denote the set of actions available at all histories $h \in \alpha_i$, and $A(\alpha)$ for the set of possible action profiles available at $\alpha \in \mathcal{AP}$. Finally, each player partitions the set of terminal histories, $Z$, into analogy classes $\zeta_i \in \Psi_i$.[5] All analogy groupings are assumed to be common knowledge.

[5]The analogy grouping of terminal histories is an attempt to capture the potential ignorance of players about (small) technical differences in outcomes. Thus, although the specific payoffs may be different, the players' perception of the different histories, as regards their strategic decisions, may not take that into account. Such a perception appears to be plausible, e.g., in many experimental settings where accumulated payoffs are converted to money at a comparably low rate. In such instances, aspects other than absolute payoffs may be more relevant to the players, e.g. the relative comparison with their opponents.

**Foresight**

Apart from the analogy grouping, each player $i$ has a foresight level $\phi_i \in \mathbb{N}$. In effect, the foresight level $\phi_i$ determines the number of stages ahead player $i$ is assumed to be aware of after any given history $h \in H$. More specifically, the convention will be that, if $\phi_i = 0$, player $i$ for each history $h$ is only informed about the set of actions available to him, $A_i(h)$, but not about who else has to move (or the actions available to these players). If, however, $\phi_i = f > 0$, then player $i$ at each $h^k \in H$ knows who else has to move at any $h^{k+l} \in H^{k+l}(h^k)$, $l = 0, \ldots, f$, the actions available to these players, i.e. $A(h^{k+l})$, as well as whether or not the game will proceed after some action-profile $a \in A(h^{k+f})$, $h^{k+f} \in H^{k+f}(h^k)$, has been chosen. Moreover, if the game proceeds, he is also informed about the analogy class $\alpha(h^{k+f} \cup \{a\})$ of the respective history; if the game ends, he is informed about the analogy class $\zeta_i$ of the respective terminal history. In Section 5.2, $\phi_i$ will become an endogenous choice variable, but for the time being, the $\phi_i$ are treated as exogenously given and are assumed to be common knowledge.

**Analogy of (Foreseen) Subtrees**

Given the players' analogy grouping and foresight, we now define the notion of analogy between (foreseen) subtrees. In order to do so, let $\Upsilon(h, \phi)$, $\phi > 0$, denote the subtree of length $\phi$ starting at $h \in H$. We say that two subtrees $\Upsilon(h, \phi)$ and $\Upsilon(\tilde{h}, \phi)$, $h \in H^k$ and $\tilde{h} \in H^m$, are analogous if the there is a mapping

$$\gamma : \bigcup_{l=0,\ldots,\phi} H^{k+l}(h) \longrightarrow \bigcup_{l=0,\ldots,\phi} H^{m+l}(\tilde{h})$$

which is one-to-one and onto, order-preserving (if $h'$ is a sub-history of $h^*$ in $\Upsilon(h, \phi)$, then so is $\gamma(h')$ for $\gamma(h^*)$ in $\Upsilon(\tilde{h}, \phi)$), and for which any two corresponding sub-histories $h'$ and $\gamma(h')$ are:

1. analogous according to the players' primary analogy grouping; i.e. for any $h' \in H^{k+l}(h)$, $0 \leq l \leq \phi$, we have: $\alpha(h') = \alpha(\gamma(h'))$, or $\zeta(h') = \zeta(\gamma(h'))$ in case of terminal histories,

2. composed of identical action profiles from the (local) origin onwards; i.e. for any $h' \in H^{k+l}(h)$, $0 < l < \phi$, we have: $a^{k+r} = \tilde{a}^{m+r}$, for all

$0 < r \leq l$, where - slightly abusing notation - $a^{k+r}$ and $\tilde{a}^{m+r}$ denote the action profiles that mark the transition between stages $k+r-1 \rightarrow k+r$ and $m+r-1 \rightarrow m+r$ according to histories $h'$ and $\gamma(h')$, respectively.[6]

We write $\Upsilon(h,\phi) \cong \Upsilon(\tilde{h},\phi)$, if $\Upsilon(h,\phi)$ and $\Upsilon(\tilde{h},\phi)$ are analogous.

**Similarity of Decisions - The Final Decision Partition**

The above defined concept of analogy between foreseen subtrees allows for a natural refinement of each player's primary analogy grouping of histories $\mathcal{AP}_i$. In particular, if $\phi > 0$, two histories $h, \tilde{h} \in \mathcal{AP}_i$ now can be disentangled by player $i$ if the foreseen subtrees are different, i.e. if $\Upsilon(h,\phi_i) \ncong \Upsilon(\tilde{h},\phi_i)$.[7] Based on the resulting partition of histories, two decisions for player $i$ are called *similar* if for the corresponding histories $h, \tilde{h} \in H$ it holds that:

1. both histories are analogous within the primary analogy grouping, i.e. $\alpha_i(h) = \alpha_i(\tilde{h})$, and
2. in case $\phi_i > 0$, the local subtrees of length $\phi_i$ with origins $h$ and $\tilde{h}$, respectively, are analogous, i.e. $\Upsilon(h,\phi) \cong \Upsilon(\tilde{h},\phi)$.

Collections of similar decisions for player $i$ are referred to as similarity classes, denoted by $\delta_i$. The set of all player $i$'s similarity classes is denoted by $\mathcal{D}_i$ and is referred to as player $i$'s *decision partition*.[8] Note that as all ingredients of $\mathcal{D}_i$, i.e. the $\mathcal{AP}_i$ and $\phi_i$, are common knowledge, so is $\mathcal{D}_i$. The set of actions available to player $i$ at similarity class $\delta_i$ is denoted by $A(\delta_i)$.

**Strategy**

A strategy for player $i$ is a function $\sigma_i$ that for each decision set $\delta_i \in \mathcal{D}_i$ specifies an action $a_i \in \Delta A(\delta_i)$, where $\Delta A(\delta_i)$ denotes the set of all mixed

[6] Note that requiring $\gamma$ to be bijective and order-preserving also implies that any two corresponding sub-histories have the same length within the respective subtree, i.e. for any $h' \in H^{k+l}(h)$, $0 \leq l \leq \phi$, we have: $\gamma(h') \in H^{m+l}(\tilde{h})$.

[7] In a sense, one might consider a player saying: *Well, essentially it is the same decision as before (e.g. to choose between public transport and a chauffeur) but the future situation I expect to be affected by my decision now is different (e.g. a business meeting now and a meeting of friends before). Therefore, the decisions are different.*

[8] As $\Upsilon(h,\phi) \cong \Upsilon(\tilde{h},\phi)$ requires $\alpha_i(h) = \alpha_i(\tilde{h})$, $\mathcal{D}_i$ is a refinement of $\mathcal{AP}_i$.

actions available at all $h_i \in \delta_i$; i.e. $\forall \delta_i$ :

$$\sigma_i : \delta_i \mapsto \sigma_{\delta_i} \in \Delta A(\delta_i).$$

Thus, $\sigma_i$ is defined as a behaviour strategy, i.e. $\sigma_i \in \times_{\delta_i \in \mathcal{D}_i} \Delta A(\delta_i)$. The set of all behaviour strategies for player $i$ is denoted by $B_i$.[9]

**Equilibrium and Limited Sequential Rationality**

As a starting point for an equilibrium concept, we consider Nash-equilibria of $\Upsilon$ with the additional restriction that for each player $i$ the (behaviour) strategy used has to be measurable with respect to the player's decision partition $\mathcal{D}_i$. Moreover, as we think of equilibrium play as being the result of an unmodelled learning process, it is desirable to impose also some minimum requirements on off-equilibrium path behaviour which is not already determined via similarity to some on-path decisions. A convenient way to do so is to require strategies to be robust to small (local) trembles in the sense of Selten (1975).[10]

In particular, a strategy profile $\sigma^* = (\sigma_i^*)_{i \in N}$, $\sigma_i^* \in B_i$, is an analogy-based limited-foresight Nash-equilibrium of $\Upsilon$, *alf-Nash-equilibrium* for short, if and only if $\sigma^*$ is the limit of some sequence of $\varepsilon$-perfect strategy profiles $\sigma^\varepsilon \in \times B_i$ for some sequence of $\varepsilon > 0$ that converges to 0. Moreover, the strategy profile $\sigma^\varepsilon \in \times B_i$ is $\varepsilon$-perfect if any induced local behaviour $\sigma^\varepsilon_{\delta_i} \in \Delta A(\delta_i)$ is completely mixed and if, for all $i$ and any pure local behaviour $a_{\delta_i} \in A_i(\delta_i)$, it holds that, if there is $\tilde{a}_{\delta_i} \in A_i(\delta_i)$ with

$$u_i(\sigma_i^\varepsilon[a_{\delta_i}], \sigma_{-i}^\varepsilon) < u_i(\sigma_i^\varepsilon[\tilde{a}_{\delta_i}], \sigma_{-i}^\varepsilon),$$

[9]Notice that the notion of behaviour strategies differs from that of mixed strategies in the present context as we do not require the similarity grouping to be compatible with perfect recall (cf. Kuhn, 1953). Yet, we adhere to behaviour strategies as we want to preclude any dependence of later actions on earlier ones which is not already accounted for in the similarity grouping.

[10]We do not build on sequential equilibria (Kreps and Wilson, 1982a) as these explicitly consider players' beliefs at unreached information sets. In the current context, however, beliefs about the actual location in the game tree are not at issue and, indeed, difficult to define, especially as long as final termination of the game is not foreseen. Nevertheless, requiring robustness of strategies to small trembles at least ensures some form of best response behaviour at decision sets that are not reached in equilibrium.

then $\sigma^{\varepsilon}_{\delta_i}(a_i) < \varepsilon$, where $\sigma^{\varepsilon}_i[a_{\delta_i}]$ denotes the behaviour strategy $\sigma_i$ for which $\sigma_{\delta_i} = a_{\delta_i}$ and $\sigma_{\delta'_i} = \sigma^{\varepsilon}_{\delta'_i}$ for all $\delta'_i \in \mathcal{D}_i$, $\delta'_i \neq \delta_i$.[11]

**Remark on Existence**
It is worth mentioning that alf-Nash equilibria, as defined above, do not exist in general. This is due to the fact that the described grouping of decision nodes almost naturally gives rise to situations of imperfect recall for which existence of a Nash-equilibrium in behaviour strategies is not guaranteed (see Wichardt, 2007, for a counterexample). In the remainder of this paper, however, existence or rather non-existence will not be an issue.

# 4 Examples

This section is devoted to a discussion of three examples to further highlight the specific effects of the alf-Nash-approach.

**Example 1 - A One Player "Game"**
To begin with, consider a single person decision problem where the only player faces a 4 times repeated choice between A and B and let us restrict attention to pure strategies. Furthermore, assume a foresight $\phi = 1$ and that no analogy grouping of terminal histories takes place. Thus, all non-terminal histories are grouped together within the primary analogy grouping and foreseen decisions are the same for the first three stages. However, the decision at the last stage is different as the player is aware of the fact that the game will terminate. Accordingly, available strategies are given by (AAA A), (AAA B), (BBB A) and (BBB B). Now if, for example, the final payoff is 0 for all patterns but (ABAB) where the payoff is 1, the player will never be able to grasp this, not even if we increase his foresight by 1 (he is still constraint to choose the same action at the first two stages).

**Example 2 - The Absent Minded Driver**
The following example is known as the absent minded driver (Piccione and Rubinstein, 1997). Its discussion shall help to clarify the relation of the

[11]This type of definition of perfect equilibria follows Myerson (1978).

present approach to earlier discussion of imperfect recall.

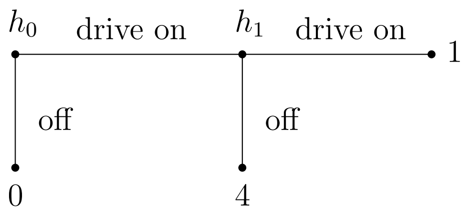


Figure 2: The 1-player game: Finding the Correct Exit on the Highway; $h_0$ and $h_1$ represent the exit decisions. With $\phi = 0$ this is the problem of the absent minded driver as introduced in Piccione and Rubinstein (1997).

The story behind the example is very simple (cf. Figure 2). A person is looking for the fastest way to drive home and the optimal way to do so is to take the second exit after entering the highway. However, all exits look very much the same. Thus, the argument goes, if the driver is absent minded, e.g. due to recent events at a bar, he may not be able to single out the correct exit. Consequently, successive exit decisions ($h_0$ and $h_1$ in Fig. 2) may be part of the same information set, thereby giving rise to a decision problem with imperfect recall. Within the alf-Nash-approach the same decision problem is generated if we assume a complete lack of foresight, e.g. for the same reasons cited above. All exits looking alike, then, would simply be analogous, and hence similar, according to the primary analogy grouping. Thus, adding to the previous discussion, the alf-Nash-approach offers a way to be more explicit about the potential sources of imperfect recall. For the sake of completeness, equilibrium in the alf-Nash framework is given by "exit with probability 1/3" (cf. Rubinstein, 1998, for a broader discussion).

**Example 3 - The Centipede Game**

As a more game theoretic example, we finally discuss how the alf-Nash-approach can be used to rationalise long periods of (not subgame perfect) passing in the Centipede Game. A general version of the Centipede Game is depicted in Figure 3 (cf. Rosenthal, 1981; the specific payoffs are taken from Aumann, 1998).

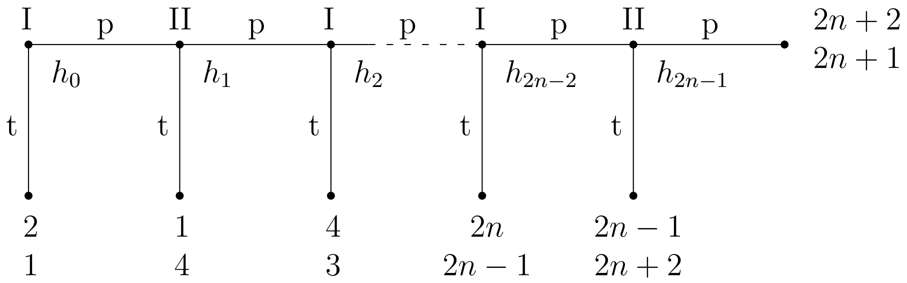


Figure 3: The Centipede Game

For the sake of argument, assume that both players group all terminal histories according to whose payoff is larger, i.e. $\Psi_1 = \Psi_2 = \{\zeta_1, \zeta_2\}$ with $\zeta_1 = \{(a_i, b_i)|i = 0, ..., 2n\}$, $\zeta_2 = \{(a_i, b_i)|i = 1, ..., 2n-1\}$, where $(a_i, b_i)$ denotes the payoffs of player 1 ($a_i$) and player 2 ($b_i$) in case of termination at $h_i$ and $(a_{2n}, b_{2n})$ denotes the respective payoffs in case of permanent passing. Moreover, assume that $\phi_I = 2$ and that $\phi_{II} = 1$. Then, for both players all their decisions are analogous according to the primary analogy grouping, but due to the foresight they both can single out their last decision with respect to the final similarity grouping; i.e. $\mathcal{D}_1 = \{\delta_1 = \{h_0, ..., h_{2n-4}\}, \tilde{\delta}_1 = \{h_{2n-2}\}\}$ and $\mathcal{D}_2 = \{\delta_2 = \{h_1, ..., h_{2n-3}\}, \tilde{\delta}_2 = \{h_{2n-1}\}\}$. Accordingly, if the game is sufficiently long, there always is an equilibrium in which both players pass in the first bit, i.e. at $\delta_1, \delta_2$, and take in the last, i.e. at $\tilde{\delta}_1, \tilde{\delta}_2$. In this case, the game is terminated at $h_{2n-2}$. There is no unravelling as the only thing player II can do is to change his behaviour for the whole set $\delta_2$ of initial decisions. For $n > 2$, however, this would lead to no improvement.

We return to the analysis of the Centipede Game in the next section when the effects of endogenous but costly foresight are discussed.

# 5 Extensions

In the following, we discuss two extensions of the alf-Nach concept introduced in Sectin 3: additional refinements of the primary analogy grouping based on (conspicuous) differences in past histories (5.1), and endogenous but costly foresight (5.2). Building on the respective extensions, it is show how less paradox equilibria in the Chain Store Game can be established (5.1), and how mixed equilibria in the Centipede Game can be obtained (5.2).

## 5.1 Incorporating Differences in Past Histories

So far, the similarity grouping of decisions is defined such that it depends solely on the players' primary analogy grouping of decisions, according to equality of available actions, and additional analogy considerations concerning foreseen sections of the game tree. In some instances, however, it seems desirable to allow also for a differentiation of decisions according to whether or not some important decision has been made in the past. If reputation is at issue, for example, it seems reasonable to expect the occurrence of some significant past behaviour to be accounted for in the similarity grouping (e.g. production of a bad quality, defection, acquiescence to market entry, etc.).

**Technical Aspects**

Additional past dependence of decisions can be accounted for by introducing an additional analogy grouping of histories, e.g. according to whether or not a certain behaviour has occurred in the past. The similarity grouping of decisions, then, is derived from the primary analogy grouping by requiring that for any two decisions of player $i$, with corresponding histories $h, \tilde{h} \in H$, it has to hold that:

1. $h$ and $\tilde{h}$ are analogous within the primary analogy grouping,
2. $h$ and $\tilde{h}$ are analogous within the analogy grouping of histories, and
3. in case $\phi_i > 0$, the local subtrees of length $\phi_i$ with origins $h$ and $\tilde{h}$, respectively, are analogous, i.e. $\Upsilon(h, \phi) \cong \Upsilon(\tilde{h}, \phi)$.

All other definitions remain the same.

**Establishing Less Paradox Equilibria in the Chain Store Game**

In order to exemplify the effects of such a history dependent similarity grouping, we apply the alf-Nash concept thus extended to the Chain Store Game

(Selten, 1978) and argue how it offers a possible resolution for the seeming paradox of this game.

*A Review*: The Chain Store Game reflects a situation in which a chain store with shops in $m$ different villages subsequently faces potential market entry of a local competitor in each of the $m$ markets, and has to decide how to react to such entry. The corresponding game can be described as follows. There are $m + 1$ players: the chain store (player A), and $m$ potential local competitors (player 1,..., player $m$), one for each village. Local competitors are assumed to enter one after the other. Thus, the game has $m$ periods or stages. In period $k$, player $k$ has to decide whether to enter (*in*) or not enter the market but invest his money elsewhere (*out*). If player $k$ chooses *in*, player A can either *fight* or *accommodate* with the new situation. If player $k$ chooses *out*, player A keeps reaping the benefits from his monopoly in market $k$. All stage game actions are observable to everyone (for a more detailed description, see Selten, 1978). Stage game payoffs are depicted in Figure 4.

| | | Player $k$: in | Player $k$: out |
|---|---|---|---|
| Player A | accom. | 2, 2 | 5, 1 |
| | fight | 0, 0 | 5, 1 |

Figure 4: Stage game payoffs for the Chain Store Game (cf. Selten, 1978).

Obviously, looking at local markets only, it is best for player A to accommodate once a local competitor has entered the market. And, in fact, all local competitors playing *in* and the chain store accommodating each entry is the unique subgame perfect equilibrium of the Chain Store Game.

The paradox in the Chain Store Game stems from the fact that, intuitively, it seems reasonable to expect aggressive responses against early entries because of the deterrent effect these may have for later periods. Following this reasoning, market entry would be accommodated only in the last few periods where the logical inconsistency of the argument is more apparent.[12] Selten in his paper describes this as the "deterrence theory" and expresses his strong affection for this type of reasoning; he writes (p. 132): *If I had to play the game in the role of Player A, I would follow the deterrence theory.*

*A Solution*: In the sequel, it shall be argued how a solution along the lines of the deterrence theory can be captured within the extended alf-Nash framework. In order to make the game amenable to the alf-Nash-approach, though, we have to assume that player A, the chain store, does not distinguish histories according to which entrant he faces. This, however, is very much in line with the structure of the game. The remaining argument is rather straight forward.

First of all, according to the primary analogy grouping, player A groups all his decisions about how to respond to market entry into one analogy class. Furthermore, assuming that no grouping of terminal histories takes place, player A will single out decisions in later periods according to the foreseen distance to the final termination of the game. Yet, if player A's foresight is smaller than the length of the game, i.e. if $\phi_A < m$, all his decisions following entry in some period $k$, $k \in \{1, \ldots, m - \phi_A\}$, remain part of the same decision set $\delta_A$. Finally, assume that all players also differentiate between histories with previous accommodated entry and those without; this seems plausible as the incumbent's reaction to entry lies at the heart of the matter. In that case, in particular player A, for stages $k$, $k \in \{1, \ldots, m-\phi_A\}$, can condition his behaviour only on whether or not previous accommodated entry has occurred. As a consequence of this, an additional, less paradox equilibrium emerges:

[12] As the termination of the game after $m$ periods is common knowledge, a standard backward induction argument proves all hopes for deterring effects to be logically unfounded.

**Proposition 1** *If player A does not distinguish decisions according to which entrant he faces, if $\phi_A$ is sufficiently small relative to $m$, and if histories with and without accommodated entry are separated in the similarity grouping, then there are two alf-Nash-equilibria for the Chain Store Game:*

*eq-1* *Player $k$, $k = 1, \ldots, m$, playing "*in*" irrespective of the history, and player A choosing "*accommodate*" against any entry.*

*eq-2* *Player $k$, $k = 1, \ldots, m - \phi_A$, playing "*in*" if some previous entry has been accommodated but playing "*out*" otherwise; player $k$, $k > m - \phi_A$, playing "*in*"; and player A choosing "*accommodate*" against entry if and only if $k > m - \phi_A$ or if he did so before (and "*fight*" otherwise).*

*Eq-1 results in the standard paradox outcome of $m$ times accommodated entry; eq-2 results in no entry in the first $m - \phi_A$ stages and accommodated entry thereafter.*

**Proof.** That eq-1 is an alf-Nash-equilibrium is immediate. Also eq-2 apparently satisfies the standard Nash-conditions. That it is also robust to small mistakes can be seen as follows. First of all, a standard argument shows that perfectness enforces *in* and *accommodate* for all stages $k$, $k > m-\phi_A$, i.e. where final termination is foreseen. Moreover, the same holds true after those histories in earlier stages with previous accommodated entry, as *accommodate* against entry here is weakly dominant for player A. Given that, however, player A playing *fight* against entry for histories $h^k$, $k \leq m - \phi_A$, with no previous accommodated entry is robust to small trembles in behaviour (if $m - \phi_A$ is large enough). In particular, accommodation of some "erroneous" entry now would lead to accommodated entry in all later periods because a new similarity class of decisions would be reached. *Fight*, however, ensures no entry (except for mistakes) in all remaining stages $k$, $k \leq m - \phi_A$, and hence is robust to small mistakes.[13] Accordingly, for $k \leq m - \phi_A$ and no previous accommodated entry, *out* is the best response for player $k$ even in case of mistakes. Hence, also eq-2 is perfect. Moreover, the above argument

[13]Note that the "stabilising" effect of the change in the similarity class after accommodated entry is a consequence of the history dependence of the similarity grouping.

shows that for all stages $k > m - \phi_A$ and all histories $h^k$, $k \leq m - \phi_A$, with previous accommodated entry, behaviour is uniquely determined by the perfectness condition. Thus, there are no further alf-Nash-equilibria for this game. ■

In our view, the second alf-Nash-equilibrium described above, to some degree, captures both, the intuition of the deterrence theory (where final termination is not foreseen/considered) as well as iterated backward reasoning (where final termination is foreseen). In that sense, the Chain Store Paradox can be resolved if the alf-Nash concept is applied.

*A Comment*: It should be noted, though, that we are by no means the first to offer a potential resolution of the chain store paradox. In particular, Kreps and Wilson (1982b, p. 254) already argue that what is missing in the standard analysis of the Chain Store Game is "a plausible mechanism that connects behavior in otherwise independent markets." They proceed to modify the game to become one of incomplete information and show how, under such conditions, the paradox can be resolved without any appeal to bounded rationality. The present approach, by contrast, offers an entirely different perspective on the problem. Relying heavily on the players' bounded rationality, it establishes the lacking link between markets through reference to rather simple analogy considerations. As also the current approach seems to capture at least part of the spirit of the original problem, we see it as an interesting complement to the discussion provided by Kreps and Wilson.

## 5.2 Endogenous but Costly Foresight

As a second extension, consider the case of an endogenous but costly choice of the foresight level. Analysing games without a predefined foresight level is appealing, for example, if little to no prior information about opponents is available to the players. If players can reasonably be expected to have some prior "idea" about each others' intricacy, as for example in the Chain Store Game discussed in the preceding subsection,[14] trying different plausible levels

[14]Eventually, the strategic aspect reflected in the game captures only part of the story.

of foresight and just checking how much they affect the equilibrium appears to be more promising. Yet, if little is known or if a situation is to be analysed more generally, endogenous but costly foresight offers a way to determine the optimal level of intricacy, in terms of foresight, under the assumption that such intricacy is costly.

**Technical Aspects**

In order to integrate endogenous but costly foresight into the alf-Nash-equilibrium concept, let $L$ denote the (maximum) length of the game $\Upsilon$ under consideration and let $c(\phi)$ denote the cost of foresight $\phi$.[15] A strategy for player $i$, then, consists of both a vector $f_i \in [0,1]^{L+1}$, $\Sigma_r f_{i,r} = 1$, specifying a probability weight for each possible foresight level $\phi \in \{0, \dots, L\}$, and for each foresight level $\phi$ a behaviour strategy $\sigma_{i,\phi} \in B_i(\phi)$.[16] In equilibrium, for each player $i$, the vectors $\sigma^*_{i,\cdot} = (\sigma^*_{i,k})_{k \in \{0,\dots,L\}}$ and $f^*_i$ have to be such that:

1. given $\phi_i = \phi$, the behaviour strategy $\sigma^*_{i,\phi}$ of player $i$ is a best response against the distribution of the other players' behaviours as induced by $(f^*_{-i}, \sigma^*_{-i,\cdot})$, i.e.

$$\sigma^*_i(\phi) \in \arg\max_{\sigma_{i,\phi}} u_i(\sigma_{i,\phi}, (f^*_{-i}, \sigma^*_{-i,\cdot}));$$

2. foresight $\phi$ is chosen by player $i$ with positive probability if and only if no other foresight offers a strictly better cost-benefit relation, i.e. $f_i(\phi) > 0$ if and only if for all $\phi' \in \{0, \dots, L\}$

$$u_i(\sigma^*_{i,\phi'}, (f^*_{-i}, \sigma^*_{-i,\cdot})) - c(\phi') \leq u_i(\sigma^*_{i,\phi}, (f^*_{-i}, \sigma^*_{-i,\cdot})) - c(\phi).$$

According to our previous definition, all strategies should also satisfy some perfectness condition; i.e. for a given $\phi_i$, player $i$'s behaviour strategy $\sigma^*_{i,\phi}$ should be robust to some small mistakes in the other players' behaviour strategies $\sigma^*_{j,\tilde{\phi}}$, $j \neq i$, $\tilde{\phi} \in \{0, \dots, L\}$. However, depending on the application,

[15]Equal costs for all players are assumed only to facilitate the ensuing discussion.

[16]By definition, available strategies in general depend on the player's foresight. Thus, with endogenous foresight, we write $B_i(\phi)$ instead of $B_i$ in order to avoid ambiguities.

the appropriate condition for sequential rationality may vary (e.g. trembles in foresight levels might be included). As the exact details are of minor importance for the subsequent example, we therefore do not specify a formal perfectness condition here in order to keep notation tractable.

*Summing up*: The main change in the concept that derives from the introduction of an endogenous but costly foresight is that it allows us to consider also randomisations over different behaviour strategies which each are connected with a different foresight level. Moreover, due to the increasing cost of higher foresight levels, there will always be a pressure towards simplification. Accordingly, many backward induction arguments may lose their force as further iteration does not just come for free. The following example exemplifies this point.

**Establishing Mixed Solutions in the Centipede Game**

Consider again the Centipede Game discussed in Example 3 of the previous section (cf. Figure 3) and assume, as before, that terminal histories are grouped according to who earns the higher payoff. Thus, players can distinguish different decisions only through increasing levels of foresight, i.e. through earlier recognition of the final termination of the game. If the end (recognised by 2 terminal nodes) is not foreseen, all future patterns look the same whenever a certain player is called upon to move. Furthermore, assume that, as described above, players also choose foresight as part of their strategy and that foresight is costly. More specifically, assume that the cost associated with a certain foresight level, $c(\phi)$, is equal for all players, with $c(0) = 0$ and $c(\phi + 1) - c(\phi) \geq c(\tilde{\phi} + 1) - c(\tilde{\phi}) > 0$ for all $\phi > \tilde{\phi}$, i.e. "units of foresight" are increasingly costly.[17]

The next proposition tells us that under such conditions there exist equilibria of the Centipede Game in which both players pass at earlier stages and start to (fully) randomise between pass and take at later ones (if at all). This result, i.e. the existence of equilibria with a phase of mixing towards the end of the game, differs from the results of almost all other approaches that can

[17] A result similar to Proposition 2 below holds true also for an idiosyncratic cost.

account for longer episodes of passing in the Centipede Game; commonly, full passing or passing up to a certain stage (mostly the last or next but last) but not mixing over different strategies can be rationalised.[18]

**Proposition 2** *Consider the Centipede Game depicted in Figure 3.*[19] *Assume that terminal histories are grouped as described above, that foresight is costly, and that $c(\phi)$ satisfies the above assumptions; then:*

1. *If $c(2) < 1$ and $n$ is sufficiently large, there exist equilibria in which both players pass in the first and randomise between pass and take in the later part of the game. The randomisation is such that that once it has started at some $h_l$, the game is terminated with probability $p(h_k)$, $0 < p(h_k) < 1$, for all $k \geq l$.*
2. *If $c(1) < 1 < c(2)$ and $n > 1$, then $\phi_1 = 0$ and $\phi_2 = 1$ followed by always pass but take at the last node is an equilibrium.*
3. *If $c(1) > 1$ and $n > 1$, then $\phi_1 = \phi_2 = 0$ and always pass is an equilibrium.*
4. *Irrespective of $c(.)$, $\phi_1 = \phi_2 = 0$ and both players playing take at their one decision set is always an equilibrium.*

**Sketch of the Proof** The intuition of the proof of the first statement is straightforward. In effect, costly foresight here is equivalent to imposing a cost on backward induction. Yet, if backward induction is costly, no player wants to invest in it if he does not actually profit from doing so – by terminating the game before the other player does. In particular, if one player played a pure strategy which amounts to termination of the game at say $h_k$, $k \leq 2n - 1$, the other player would either want to foresee final termination already at $h_{k-1}$ and terminate the game there or, and that is the crucial difference to the standard backward induction procedure, would want to save

[18]Other approaches are, for example, due to Kreps et al. (1982), Neyman (1985, 1998), or Jehiel (2004). See Jehiel (2004, pp. 91-97) for a review.

[19]The assumption of linearly increasing payoffs (cf. Figure 3) is not restrictive. Similar results can be shown to hold for exponentially increasing payoffs as used in the experimental study by McKelvey and Palfrey (1992).

the cost of foresight and always pass.[20] Now, if nobody has any foresight, i.e. for $\phi_1 = \phi_2 = 0$, it is obviously optimal for player 2 to invest in $\phi_2 = 1$ and to terminate the game at $h_{2n-1}$, the last node, as $c(2) < 1$. Given that, however, it is optimal for player 1 to himself "buy" $\phi_1 = 2$ and terminate the game at $h_{2n-2}$, in order to pre-empt player 2; and so forth. As units of foresight are increasingly costly, this "race to the bottom" eventually will come to an end; i.e. at some stage further units of foresight will become prohibitively costly so that the respective player will rather return to $\phi = 0$ and always pass. The resulting situation, however, cannot be an equilibrium either as the remaining player now is left with an inefficient level of foresight ($\phi_i \geq 2$). Hence, what results are the type of mixed equilibria described in Proposition 2.1. All other statements are immediate. q.e.d.[21]

*Two Comments*: First, we want to comment on the possible interpretations of the randomisation over foresight levels. These we think are at least twofold. On the one hand, one can of course think of all players being equal and choosing each time randomly how much effort - in terms of foresight - they want to devote to the current play of the game. On the other hand, however, it is also possible, and perhaps somewhat more plausible, to appeal to the common population average interpretation of randomisation. In the present case, this would imply that the players in the population may differ in their degree of intricacy - measured by the ability to foresee future stages of the game. And the use of different foresight levels in equilibrium would be interpreted as an instance where higher intricacy does not pay beyond its cost.

Second and finally, it is interesting to note that in the mixed equilibrium described in Proposition 2 players randomise only over foresight levels but choose pure strategies for any given level of foresight. However, it should be clear that this is only an artefact of the special strategic structure of the Centipede Game. In general, players may well use behaviour strategies that

[20] If $k < 4$, then $\phi = 0$ and $t$ might be the best response. However, this will not occur if $n$ is large.

[21] A complete proof of the statement is available from the author on request.

involve proper randomisation for a given positive foresight level. It only has to be the case that none of the actions involved in the randomisation for a given foresight level is also available with less foresight as that would always be cheaper.

# 6 Concluding Remarks

In this paper, we have proposed a bounded rationality approach to model equilibrium play in games. The approach taken is based on the assumption that (real) decision makers are often inclined to simplify decision processes and, therefore, tend to take the same actions in similar situations. Similarity in our context is determined through the players' general analogy grouping of decisions (according to equality of available actions) and their ability to distinguish between different future patterns of the game tree - if foreseen.

In the preceding sections, we have exemplified the benefits of the alf-Nash approach discussing various different examples as well as possible extensions. We have argued how the famous Chain Store Paradox can be resolve within the present framework if we also account for an analogy grouping of histories according to past actions. And it was shown how mixed equilibria in the Centipede Game can be obtained if foresight becomes a costly choice variable.

Summing up, the alf-Nash approach emphasises a new perspective on individual decision making, which enables us to capture a variety of otherwise seemingly irrational behavioural patterns in a technical equilibrium concept. As a modelling device, it allows us to analyse decision processes under various plausible (situation specific) assumptions regarding the respective players' rationality, i.e. foresight levels and analogy considerations. And although equilibrium play in some cases may be very sensitive to the exact specification of these assumptions, there are many instances in which this dependence is less strong or almost negligible (cf. the examples discussed in this paper). In fact, we see it as one of the advantages of the alf-Nash approach that it allows us to become aware of, pin down and study these differences. Thus, we are confident that it will prove helpful in the attempt to reconcile observed behaviour with the game theoretic predictions and to gain a more profound understanding of (not always rational) economic decision making.

**References**

Aumann, R., 1998, "Note on the Centipede Game," Games and Economic Behavior 23, pp. 97-105.

Camerer, C., 2003, *Behavioral Game Theory*, Princeton University Press, Princeton, New Jersey.

Fudenberg, D., and J. Tirole, 1991, *Game Theory*, MIT Press, Cambridge, Massachusetts.

Heifetz, A., M. Meier and B. Schipper, 2006, "Interactive Unawareness," Journal of Economic Theory 130, pp. 78-94.

Jehiel, P., 1995, "Limited Horizon Forecast in Repeated Alternate Games," Journal of Economic Theory 67, pp. 479-519.

Jehiel, P., 1998, "Learning to Play Limited Forecast Equilibria," Games and Economic Behavior 22, pp. 274-298.

Jehiel, P., 2001, "Limited Foresight May Force Cooperation," Review of Economic Studies 68, pp. 369-391.

Jehiel, P., 2005, "Analogy-Based Expectation Equilibrium," Journal of Economic Theory 123, pp. 81-104.

Jehiel, P., and D. Samet, 2005, "Learning to Play Games in Extensive Form by Valuation," Journal of Economic Theory 124, pp. 129-148.

Kreps, D., P. Milgrom, J. Roberts and R. Wilson, 1982, "Rational Cooperation in the Finitely Repeated Prisoners' Dilemma," Journal of Economic Theory 27, pp. 245-252.

Kreps, D., and R. Wilson, 1982a, "Sequential Equilibrium," Econometrica 50, pp. 863-894.

Kreps, D., and R. Wilson, 1982b, "Reputation and Imperfect Information," Journal of Economic Theory 27, pp. 253-279.

Kuhn, H.W., 1953, "Extensive Games and the Problem of Information," pp. 193-216 in *Contributions to the Theory of Games II*, Princeton University Press, Princeton, New Jersey.

McKelvey, R., and T. Palfrey, 1992, "An Experimental Study of the Centipede Game," Econometrica 60, pp. 803-836.

Myerson, R., 1978, "Refinements of the Nash Equilibrium Concept," International Journal of Game Theory 7, pp. 73-80.

Neyman, A., 1985, "Bounded Complexity Justifies Cooperation in the Finitely Repeated Prisoners' Dilemma," Economics Letters 19, pp. 227-229.

Neyman, A., 1998, "Finitely Repeated Games with finite Automata," Mathematics of Operations Research 23, pp. 513-552.

Piccione, M., and A. Rubinstein, 1997, "On the Interpretation of Decision Problems with Imperfect Recall," Games and Economic Behavior 20, pp. 3-24.

Rosenthal, R., 1981, "Games of Perfect Information, Predatory Pricing and the Chain-Store Paradox," Journal of Economic Theory 25, pp. 92-100.

Rubinstein, A., 1986, "Finite Automata Play a Repeated Prisoner's Dilemma," Journal of Economic Theory 39, pp. 83-96.

Rubinstein, A. 1998, *Modeling Bounded Rationality*, MIT Press, Cambridge, Massachusetts.

Selten, R., 1975, "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory 4, pp. 25-55.

Selten, R., 1978, "The Chain Store Paradox," Theory and Decision 9, pp. 127-159.

Wichardt, P., 2007, "Existence of Nash Equilibria in Finite Extensive Form Games with Imperfect Recall: A Counterexample," Games and Economic Behavior, article in press.